Undulating Relativity

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ABSTRACT

The Special Theory of Relativity takes us to two results that presently are considered “inexplicable” to many renowned scientists, to know:

-The dilatation of time, and
-The contraction of the Lorentz Length.

The solution to these have driven the author to the development of the Undulating Relativity (UR) theory, where the Temporal variation is due to the differences on the route of the light propagation and the lengths are constants between two landmarks in uniform relative movement.

The Undulating Relativity provides transformations between the two landmarks that differs from the transformations of Lorentz for: Space (x,y,z), Time (t), Speed ($\vec{u}$), Acceleration ($\vec{a}$), Energy (E), Momentum ($\vec{p}$), Force ($\vec{F}$), Electrical Field ($\vec{E}$), Magnetic Field ($\vec{B}$), Light Frequency (y), Electrical Current ($\vec{J}$) and “Electrical Charge” (ρ).

From the analysis of the development of the Undulating Relativity, the following can be synthesized:

- It is a theory with principles completely on physics;
- The transformations are linear;
- Keeps untouched the Euclidian principles;
- Considers the Galileo’s transformation distinct on each referential;
- Ties the Speed of Light and Time to a unique phenomenon;
- The Lorentz force can be attained by two distinct types of Filed Forces, and
- With the absence of the spatial contraction of Lorentz, to reach the same classical results of the special relativity rounding is not necessary as concluded on the Doppler effect.

Both, the Undulating Relativity and the Special Relativity of Albert Einstein explain the experience of Michel-Morley, the longitudinal and transversal Doppler effect, and supplies exactly identical formulation to:

Aberration of zenith $\Rightarrow tg\alpha = \frac{v}{c} / \sqrt{1 - \frac{v^2}{c^2}}$.

Fresnel’s formula $\Rightarrow c' = \frac{c}{n} + v(1 - \frac{1}{n^2})$.

Mass (m) with velocity (v) = [resting mass (m_o)]/ $\sqrt{1 - \frac{v^2}{c^2}}$.

$E = mc^2$.

Momentum $\Rightarrow \vec{p} = \frac{m_o \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$.

Relation between momentum (p) and Energy (E) $\Rightarrow E = c.\sqrt{m_o^2.c^2 + p^2}$.

Relation between the electric field ($\vec{E}$) and the magnetic field ($\vec{B}$) $\Rightarrow \vec{B} = \frac{\vec{V}}{c^2} \times \vec{E}$.

Biot-Savant’s formula $\Rightarrow \vec{B} = \frac{\mu_o I}{2\pi.R} \vec{u}$.

Louis De Broglie’s wave equation $\Rightarrow \psi(x,t) = a.sin\left[2\pi\gamma\left(t - \frac{x}{u}\right)\right]; u = \frac{c^2}{v}$

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Undulating Relativity

§ 1 Transformation to space and time

The Undulating Relativity (UR) keep the principle of the relativity and the principle of Constancy of light speed, exactly like Albert Einstein's Special Relativity Theory defined:

a) The laws, under which the state of physics systems are changed are the same, either when referred to a determined system of coordinates or to any other that has uniform translation movement in relation to the first.

b) Any ray of light moves in the resting coordinates system with a determined velocity c, that is the same, whatever this ray is emitted by a resting body or by a body in movement (which explains the experience of Michel-Morley).

Let's imagine first that two observers O and O' (in vacuum), moving in uniform translation movement in relation to each other, that is, the observer don't rotate relatively to each other. In this way, the observer O together with the axis x, y, and z of a system of a rectangle Cartesian coordinates, sees the observer O' move with velocity v, on the positive axis x, with the respective parallel axis and sliding along with the x axis while the O', together with the x', y' and z' axis of a system of a rectangle Cartesian coordinates sees O moving with velocity –v', in negative direction towards the x' axis with the respective parallel axis and sliding along with the x' axis. The observer O measures the time t and the O' observer measures the time t' (t ≠ t'). Let's admit that both observers set their clocks in such a way that, when the coincidence of the origin of the coordinated system happens t = t' = zero.

In the instant that t = t' = 0, a ray of light is projected from the common origin to both observers. After the time interval t the observer O will notice that his ray of light had simultaneously hit the coordinates point A (x, y, z) with the ray of the O' observer with velocity c and that the origin of the system of the O' observer has run the distance v t along the positive way of the x axis, concluding that:

$$x^2 + y^2 + z^2 - c^2 t^2 = 0 \qquad 1.1$$

$$x' = x - v t. \qquad 1.2$$

The same way after the time interval t' the O' observer will notice that his ray of light simultaneously hit with the observer O the coordinate point A (x', y', z') with velocity c and that the origin of the system for the observer O has run the distance v't' on the negative way of the axis x', concluding that:

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0 \qquad 1.3$$

$$x = x' + v' t'. \qquad 1.4$$

Making 1.1 equal to 1.3 we have

$$x^2 + y^2 + z^2 - c^2 t^2 = x'^2 + y'^2 + z'^2 - c^2 t'^2. \qquad 1.5$$

Because of the symmetry y = y' end z = z', that simplify 1.5 in

$$x^2 - c^2 t^2 = x'^2 - c^2 t'^2. \qquad 1.6$$

To the observer O x' = x – v t (1.2) that applied in 1.6 supplies

$x^2 - c^2 t^2 = (x - v t)^2 - c^2 t'^2$ from where

$$t' = t\sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2 t}}. \qquad 1.7$$

To the observer O' x = x' + v' t' (1.4) that applied in 1.6 supplies

$(x' + v' t')^2 - c^2 t^2 = x'^2 - c^2 t'^2$ from where

$$t = t'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}}\ .$$ 1.8

Table I, transformations to the space and time

x' = x – v t	1.2	x = x' + v' t'	1.4
y' = y	1.2.1	y = y'	1.4.1
z' = z	1.2.2	z = z'	1.4.2
$t' = t\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}}$	1.7	$t = t'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}}$	1.8

From the equation system formed by 1.2 and 1.4 we find

v t = v' t' or $|v|t = |v'|t'$ (considering t>o e t'>0) 1.9

what demonstrates the invariance of the space in the Undulatory Relatitivy.

From the equation system formed by 1.7 and 1.8 we find

$$\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}}\cdot\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}} = 1.$$ 1.10

If in 1.2 x' = 0 then x = v t, that applied in 1.10 supplies,

$$\sqrt{1-\frac{v^2}{c^2}}\cdot\sqrt{1+\frac{v'^2}{c^2}} = 1.$$ 1.11

If in 1.10 x = ct and x' = c t' then

$$\left(1-\frac{v}{c}\right).\left(1+\frac{v'}{c}\right)=1.$$ 1.12

To the observer O the principle of light speed constancy guarantees that the components ux, uy and uz of the light speed are also constant along its axis, thus

$$\frac{x}{t}=\frac{dx}{dt}=ux,\frac{y}{t}=\frac{dy}{dt}=uy,\frac{z}{t}=\frac{dz}{dt}=uz$$ 1.13

and then we can write

$$\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}} = \sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}.$$ 1.14

With the use of 1.7 and 1.9 and 1.14 we can write

$$\frac{|v|}{|v'|}=\frac{t'}{t} = \sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}} = \sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}.$$ 1.15

Differentiating 1.9 with constant v and v', or else, only the time varying we have

$|v|dt = |v'|dt'$ or $\frac{|v|}{|v'|}=\frac{dt'}{dt}$, 1.16

but from 1.15 $\frac{|v|}{|v'|}=\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}$ then $dt' = dt\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}$. 1.17

Being v and v' constants, the reazons $\frac{|v|}{|v'|}$ and $\frac{t'}{t}$ in 1.15 must also be constant because fo this the differential of $\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}}$ must be equal to zero from where we conclude $\frac{x}{t}=\frac{dx}{dt}=ux$, that is exactly the same as 1.13.

To the observer O' the principle of Constancy of velocity of light guarantees that the components u'x', u'y', and u'z' of velocity of light are also constant alongside its axis, thus

$$\frac{x'}{t'}=\frac{dx'}{dt'}=u'x',\frac{y'}{t'}=\frac{dy'}{dt'}=u'y',\frac{z'}{t'}=\frac{dz'}{dt'}=u'z', \qquad 1.18$$

and with this we can write ,

$$\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}} = \sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}. \qquad 1.19$$

With the use of 1.8, 1.9, and 1.19 we can write

$$\frac{|v'|}{|v|}=\frac{t}{t'} = \sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}} = \sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}. \qquad 1.20$$

Differentiating 1.9 with v' and v constant, that is, only the time varying we have

$$|v'|dt'=|v|dt \text{ or } \frac{|v'|}{|v|}=\frac{dt}{dt'}, \qquad 1.21$$

but from 1.20 $\frac{|v'|}{|v|}=\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}$ then $dt=dt'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}$. 1.22

Being v' and v constant the divisions $\frac{|v'|}{|v|}$ and $\frac{t}{t'}$ in 1.20 also have to be constant because of this the differential of $\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}}$ must be equal to zero from where we conclude $\frac{x'}{t'}=\frac{dx'}{dt'}=u'x'$, that is exactly like to 1.18.

Replacing 1.14 and 1.19 in 1.10 we have

$$\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}.\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}} = 1. \qquad 1.23$$

To the observer O the vector position of the point A of coordinates (x,y,z) is

$$\vec{R}=x\vec{i}+y\vec{j}+z\vec{k}, \qquad 1.24$$

and the vector position of the origin of the system of the observer O' is

$$\vec{R}o'=vt\vec{i}+0\vec{j}+0\vec{k}\Rightarrow\vec{R}o'=vt\vec{i}. \qquad 1.25$$

To the observer O', the vector position of the point A of coordinates (x',y',z') is

$$\vec{R}'=x'\vec{i}+y'\vec{j}+z'\vec{k}, \qquad 1.26$$

and the vector position of the origin of the system of the observer O is

$$\vec{R}o=-v't'\vec{i}+0\vec{j}+0\vec{k}\Rightarrow\vec{R}o=-v't'\vec{i}. \qquad 1.27$$

Due to 1.9, 1.25, and 1.27 we have, $\vec{R}o'=-\vec{R}o$. 1.28

As 1.24 is equal to 1.25 plus 1.26 we have

$$\vec{R}=\vec{R}o'+\vec{R}'\Rightarrow\vec{R}'=\vec{R}-\vec{R}o'. \qquad 1.29$$

Applying 1.28 in 1.29 we have, $\vec{R}=\vec{R}'-\vec{R}o$. 1.30

To the observer O the vector velocity of the origin of the system of the observer O' is

$$\vec{v}=\frac{d\vec{R}o'}{dt}=v\vec{i}+0\vec{j}+0\vec{k}\Rightarrow\vec{v}=v\vec{i}\,. \qquad 1.31$$

To the observer O' the vector velocity of the origin of the system of the observer O is

$$\vec{v}'=\frac{d\vec{R}o}{dt}=-v'\vec{i}+0\vec{j}+0\vec{k}\Rightarrow\vec{v}'=-v'\vec{i}\,. \qquad 1.32$$

From 1.15, 1.20, 1.31, and 1.32 we find the following relations between $\vec{v}$ and $\vec{v}'$

$$\vec{v}=\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}} \qquad 1.33$$

$$\vec{v}'=\frac{-\vec{v}}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}}\,. \qquad 1.34$$

Observation: in the table I the formulas 1.2, 1.2.1, and 1.2.2 are the components of the vector 1.29 and the formulas 1.4, 1.4.1, and 1.4.2 are the components of the vector 1.30.

§2 Law of velocity transformations $\vec{u}$ and $\vec{u}'$

Differentiating 1.29 and dividing it by 1.17 we have

$$\frac{d\vec{R}'}{dt'}=\frac{d\vec{R}-d\vec{R}o'}{dt\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}}\Rightarrow\vec{u}'=\frac{\vec{u}-\vec{v}}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}}=\frac{\vec{u}-\vec{v}}{\sqrt{K}}\,. \qquad 2.1$$

Differentiating 1.30 and dividing it by 1.22 we have

$$\frac{d\vec{R}}{dt}=\frac{d\vec{R}'-d\vec{R}o}{dt'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}}\Rightarrow\vec{u}=\frac{\vec{u}'-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}}=\frac{\vec{u}'-\vec{v}'}{\sqrt{K'}}\,. \qquad 2.2$$

Table 2, Law of velocity transformations $\vec{u}$ and $\vec{u}'$

$\vec{u}'=\frac{\vec{u}-\vec{v}}{\sqrt{K}}$	2.1	$\vec{u}=\frac{\vec{u}'-\vec{v}'}{\sqrt{K'}}$	2.2
$u'x'=\frac{ux-v}{\sqrt{K}}$	2.3	$ux=\frac{u'x'+v'}{\sqrt{K'}}$	2.4
$u'y'=\frac{uy}{\sqrt{K}}$	2.3.1	$uy=\frac{u'y'}{\sqrt{K'}}$	2.4.1
$u'z'=\frac{uz}{\sqrt{K}}$	2.3.2	$uz=\frac{u'z'}{\sqrt{K'}}$	2.4.2
$\lvert v'\rvert=\frac{\lvert v\rvert}{\sqrt{K}}$	1.15	$v=\frac{\lvert v'\rvert}{\sqrt{K'}}$	1.20
$\sqrt{K}=\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}$	2.5	$\sqrt{K'}=\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}$	2.6

Multiplying 2.1 by itself we have

$$u'=\frac{u\sqrt{1+\frac{v^2}{u^2}-\frac{2vux}{u^2}}}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}}. \qquad 2.7$$

If in 2.7 we make u = c then u' = c as it is required by the principle of constancy of velocity of light.
Multiplying 2.2 by itself we have

$$u=\frac{u'\sqrt{1+\frac{v'^2}{u'^2}+\frac{2v'u'x'}{u'^2}}}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}}. \qquad 2.8$$

If in 2.8 we make u' = c then u = c as it is required by the principle of constancy of velocity of light.

If in 2.3 we make ux = c then $u'x'=\frac{c-v}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vc}{c^2}}}=c$ as it is required by the principle of constancy of velocity of light.

If in 2.4 we make u'x' = c then $ux=\frac{c+v'}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'c}{c^2}}}=c$ as it is required by the principle of constancy of velocity of light.

Remodeling 2.7 and 2.8 we have

$$\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}=\frac{\sqrt{1-\frac{u^2}{c^2}}}{\sqrt{1-\frac{u'^2}{c^2}}}. \qquad 2.9$$

$$\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}=\frac{\sqrt{1-\frac{u'^2}{c^2}}}{\sqrt{1-\frac{u^2}{c^2}}}. \qquad 2.10$$

The direct relations between the times and velocities of two points in space can be obtained with the equalities $u'=0\Rightarrow u'x'=0\Rightarrow ux=v$ coming from 2.1, that applied in 1.17, 1.22, 1.20, and 1.15 supply

$$dt'=dt\sqrt{1+\frac{v^2}{c^2}-\frac{2vv}{c^2}}\Rightarrow dt=\frac{dt'}{\sqrt{1-\frac{v^2}{c^2}}}, \qquad 2.11$$

$$dt=dt'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'0}{c^2}}\Rightarrow dt'=\frac{dt}{\sqrt{1+\frac{v'^2}{c^2}}}, \qquad 2.12$$

$$|v|=\frac{|v'|}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'0}{c^2}}}\Rightarrow|v|=\frac{|v'|}{\sqrt{1+\frac{v'^2}{c^2}}}, \qquad 2.13$$

$$|v'|=\frac{|v|}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vv}{c^2}}}\Rightarrow|v'|=\frac{|v|}{\sqrt{1-\frac{v^2}{c^2}}}. \qquad 2.14$$

Aberration of the zenith

To the observer O' along with the star u'x' = 0, u'y' = c and u'z' = 0, and to the observer O along with the Earth we have the conjunct 2.3

$$0=\frac{ux-v}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}}\Rightarrow ux=v,\ c=\frac{uy}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vv}{c^2}}}\Rightarrow uy=c\sqrt{1-\frac{v^2}{c^2}}$$, uz = 0,

$$u=\sqrt{ux^2+uy^2+uz^2}=\sqrt{v^2+\left(c\sqrt{1-\frac{v^2}{c^2}}\right)^2+0^2}=c$$ exactly as foreseen by the principle of relativity.

To the observer O the light propagates in a direction that makes an angle with the vertical axis y given by

$$tang\alpha=\frac{ux}{uy}=\frac{v}{c.\sqrt{1-\frac{v^2}{c^2}}}=\frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}} \qquad 2.15$$

that is the aberration formula of the zenith in the special relativity .
If we inverted the observers we would have the conjunct 2.4

$$0=\frac{u'x'+v'}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}}\Rightarrow u'x'=-v',\ c=\frac{u'y'}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'(-v')}{c^2}}}\Rightarrow u'y'=c\sqrt{1-\frac{v'^2}{c^2}}$$, u'z'=0,

$$u'=\sqrt{u'x'^2+u'y'^2+u'z'^2}=\sqrt{(-v')^2+\left(c\sqrt{1-\frac{v'^2}{c^2}}\right)^2+0^2}=c$$

$$tang\alpha=\frac{u'x'}{u'y'}=\frac{-v'}{c.\sqrt{1-\frac{v'^2}{c^2}}}=\frac{-v'/c}{\sqrt{1-\frac{v'^2}{c^2}}} \qquad 2.16$$

that is equal to 2.15, with the negative sign indicating the contrary direction of the angles.

Fresnel's formula

Considering in 2.4, $u'x'=c/n$ the velocity of light relativily to the water, $v'=v$ the velocity of water in relation to the apparatus then $ux=c'$ will be the velocity of light relatively to the laboratory

$$c'=\frac{c/n+v}{\sqrt{1+\frac{v^2}{c^2}+\frac{2vc/n}{c^2}}}=\frac{c/n+v}{\sqrt{1+\frac{v^2}{c^2}+\frac{2v}{nc}}}=\left(\frac{c}{n}+v\right)\left(1+\frac{v^2}{c^2}+\frac{2v}{nc}\right)^{-\frac{1}{2}}\cong\left(\frac{c}{n}+v\right)\left[1-\frac{1}{2}\left(\frac{v^2}{c^2}+\frac{2v}{nc}\right)\right]$$

Ignoring the term v^2/c^2 we have

$$c'\cong\left(\frac{c}{n}+v\right)\left(1-\frac{v}{nc}\right)\cong\frac{c}{n}+v-\frac{v}{n^2}-\frac{v^2}{nc}$$

and ignoring the term v^2/nc we have the Fresnel's formula

$$c'=\frac{c}{n}+v-\frac{v}{n^2}=\frac{c}{n}+v\left(1-\frac{1}{n^2}\right). \qquad 2.17$$

Doppler effect

Making $r^2 = x^2 + y^2 + z^2$ and $r'^2 = x'^2 + y'^2 + z'^2$ in 1.5 we have $r^2 - c^2t^2 = r'^2 - c^2t'^2$ or $(r-ct) = (r'-ct')\frac{(r'+ct')}{(r+ct)}$ replacing then $r = ct$, $r' = ct'$ and 1.7 we find $(r-ct) = (r'-ct')\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}}$ as $c = \frac{w}{k} = \frac{w'}{k'}$ then $\frac{1}{k}(kr-wt) = \frac{1}{k'}(k'r'-w't')\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}}$ where to attend the principle of relativity we will define $k' = k\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}}$ 2.18

Resulting in the expression $(kr-wt) = (k'r'-w't')$ symmetric and invariable between the observers.

To the observer O an expression in the formula of $\psi(r,t) = f(kr-wt)$ 2.19

represents a curve that propagates in the direction of $\vec{R}$. To the observer O' an expression in the formula of

$\psi(r',t') = f'(k'r'-w't')$ 2.20

represents a curve that propates in the direction of $\vec{R}$.

Applying in 2.18 $k = \frac{2\pi}{\lambda}$, $k' = \frac{2\pi}{\lambda'}$, 1.14, 1.19, 1.23, 2.5, and 2.6 we have

$\lambda' = \frac{\lambda}{\sqrt{K}}$ e $\lambda = \frac{\lambda'}{\sqrt{K'}}$, 2.21

that applied in $c = y\lambda = y'\lambda'$ supply, $y' = y\sqrt{K}$ and $y = y'\sqrt{K'}$. 2.22

Considering the relation of Planck-Einstein between energy (E) and frequency (y), we have to the observer O $E = hy$ and to the observer O' $E' = hy'$ that replaced in 2.22 supply

$E' = E\sqrt{K}$ and $E = E'\sqrt{K'}$. 2.23

If the observer O that sees the observer O' moving with velocity v in a positive way to the axis x, emits waves of frequency y and velocity c in a positive way to the axis x then, according to 2.22 and $ux = c$ the observer O' will measure the waves with velocity c and frequency $y' = y\left(1-\frac{v}{c}\right)$, 2.24

that is exactly the classic formula of the longitudinal Doppler effect.

If the observer O' that sees the observer O moving with velocity –v' in the negative way of the axis x', emits waves of frequency y' and velocity c, then the observer O according to 2.22 and $u'x' = -v'$ will measure waves of frequency y and velocity c in a perpendicular plane to the movement of O' given by

$\gamma = \gamma'\sqrt{1-\frac{v'^2}{c^2}}$, 2.25

that is exactly the formula of the transversal Doppler effect in the Special Relativity.

§3 Transformations of the accelerations $\vec{a}$ and $\vec{a}'$

Differentiating 2.1 and dividing it by 1.17 we have

$$\frac{du'}{dt'} = \frac{du/\sqrt{K}}{dt\sqrt{K}} + (u-v)\frac{v}{c^2}\frac{dux/K\sqrt{K}}{dt\sqrt{K}} \Rightarrow a' = \frac{a}{K} + (u-v)\frac{v}{c^2}\frac{ax}{K^2}. \quad 3.1$$

Differentiating 2.2 and dividing it by 1.22 we have

$$\frac{du}{dt} = \frac{du'/\sqrt{K'}}{dt'\sqrt{K'}} - (u'-v')\frac{v'}{c^2}\frac{du'x'/K'\sqrt{K'}}{dt'\sqrt{K'}} \Rightarrow a = \frac{a'}{K'} - (u'-v')\frac{v'}{c^2}\frac{a'x'}{K'^2}. \quad 3.2$$

Table 3, transformations of the accelerations $\vec{a}$ and $\vec{a}'$

$\vec{a}'=\frac{\vec{a}}{K}+(\vec{u}-\vec{v})\frac{v}{c^2}\frac{ax}{K^2}$	3.1	$\vec{a}=\frac{\vec{a}'}{K'}-(\vec{u}'-\vec{v}')\frac{v'}{c^2}\frac{a'x'}{K'^2}$	3.2
$a'x'=\frac{ax}{K}+(ux-v)\frac{v}{c^2}\frac{ax}{K^2}$	3.3	$ax=\frac{a'x'}{K'}-(u'x'+v')\frac{v'}{c^2}\frac{a'x'}{K'^2}$	3.4
$a'y'=\frac{ay}{K}+uy\frac{v}{c^2}\frac{ax}{K^2}$	3.3.1	$ay=\frac{a'y'}{K'}-u'y'\frac{v'}{c^2}\frac{a'x'}{K'^2}$	3.4.1
$a'z'=\frac{az}{K}+uz\frac{v}{c^2}\frac{ax}{K^2}$	3.3.2	$az=\frac{a'z'}{K'}-u'z'\frac{v'}{c^2}\frac{a'x'}{K'^2}$	3.4.2
$a'=\frac{a}{K}$	3.8	$a=\frac{a'}{K'}$	3.9
$K=1+\frac{v^2}{c^2}-\frac{2vux}{c^2}$	3.5	$K'=1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}$	3.6

From the tables 2 and 3 we can conclude that if to the observer O $\vec{u}.\vec{a}=zero$ and $c^2=ux^2+uy^2+uz^2$, then it is also to the observer O' $\vec{u}'.\vec{a}'=zero$ and $c^2=u'x'^2+u'y'^2+u'z'^2$, thus $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u}'$ is perpendicular to $\vec{a}'$ as the vectors theory requires.

Differentiating 1.9 with the velocities and the times changing we have, $tdv+vdt=t'dv'+v'dt$, but considering 1.16 we have, $vdt=v'dt\Rightarrow tdv=t'dv'$ 3.7

Where replacing 1.15 and dividing it by 1.17 we have, $\frac{dv'}{dt}=\frac{dv}{dtK}$ or $a'=\frac{a}{K}$. 3.8

We can also replace 1.20 in 3.7 and divide it by 1.22 deducing
$\frac{dv}{dt}=\frac{dv'}{dtK'}$ or $a=\frac{a'}{K'}$. 3.9

The direct relations between the modules of the accelerations a and a' of two points in space can be obtained with the $\vec{u}'=0\Rightarrow u'x'=0\Rightarrow a'x'=0\Rightarrow\vec{u}=\vec{v}\Rightarrow ux=v$ coming from 2.1, that applied in 3.8 and 3.9 supply

$$a'=\frac{a}{1+\frac{v^2}{c^2}-\frac{2vv}{c^2}}=\frac{a}{1-\frac{v^2}{c^2}} \text{ and } a=\frac{a'}{1+\frac{v'^2}{c^2}+\frac{2v'0}{c^2}}=\frac{a'}{1+\frac{v'^2}{c^2}}. \quad 3.10$$

That can also be reduced from 3.1 and 3.2 if we use the same equalities

$\vec{u}'=0\Rightarrow u'x'=0\Rightarrow a'x'=0\Rightarrow\vec{u}=\vec{v}\Rightarrow ux=v$ coming from 2.1.

§4 Transformations of the Moments $\vec{p}$ and $\vec{p}'$

Defined as $\vec{p}=m(u)\vec{u}$ and $\vec{p}'=m'(u')\vec{u}'$, 4.1

where $m(u)$ and $m'(u')$ symbolizes the function masses of the modules of velocities $u=|\vec{u}|$ and $u'=|\vec{u}'|$.

We will have the relations between $m(u)$ and $m'(u')$ and the resting mass m_o, analyzing the elastic collision in a plane between the sphere s that for the observer o moves alongside the axis y with velocity uy = w and the sphere s' that for the observer O' moves alongside the axis y' with velocity u'y' = -w. The spheres while observed in relative resting are identical and have the mass m_o. The considered collision is symmetric in relation to a parallel line to the axis y and y' passing by the center of the spheres in the moment of. Collision.

Before and after the collision the spheres have velocities observed by O and O' according to the following table gotten from table 2

	Sphere	Observer O	Observer O’
Before	s	$u_{xs}=zero,\ u_{ys}=w$	$u'_{x's}=-v,\ u'_{y's}=w\sqrt{1-\frac{v^2}{c^2}}$
Collision	s’	$u_{xs'}=v,\ u_{ys'}=-w\sqrt{1-\frac{v^2}{c^2}}$	$u'_{x's'}=zero,\ u'_{y's'}=-w$
After	s	$u_{xs}=zero,\ u_{ys}=-\overline{w}$	$u'_{x's}=-v,\ u'_{y's}=-\overline{w}\sqrt{1-\frac{v^2}{c^2}}$
Collision	s’	$u_{xs'}=v,\ u_{ys'}=\overline{w}\sqrt{1-\frac{v^2}{c^2}}$	$u'_{x's'}=zero,\ u'_{y's'}=\overline{w}$

To the observer O, the principle of conservation of moments establishes that the moments $p_x=m(u)u_x$ and $p_y=m(u)u_y$, of the spheres s and s’ in relation to the axis x and y, remain constant before and after the collision thus for the axis x we have

$$m\left(\sqrt{u_{xs}^2+u_{ys}^2}\right)u_{xs}+m\left(\sqrt{u_{xs'}^2+u_{ys'}^2}\right)u_{xs'}=m\left(\sqrt{u_{xs}^2+u_{ys}^2}\right)u_{xs}+m\left(\sqrt{u_{xs'}^2+u_{ys'}^2}\right)u_{xs},$$

where replacing the values of the table we have

$$m\left(\sqrt{v^2+\left(-w\sqrt{1-\frac{v^2}{c^2}}\right)^2}\right)v=m\left(\sqrt{v^2+\left(\overline{w}\sqrt{1-\frac{v^2}{c^2}}\right)^2}\right)v$$ from where we conclude that $\overline{w}=w$,

and for the axis y

$$m\left(\sqrt{u_{xs}^2+u_{ys}^2}\right)u_{ys}+m\left(\sqrt{u_{xs'}^2+u_{ys'}^2}\right)u_{ys'}=m\left(\sqrt{u_{xs}^2+u_{ys}^2}\right)u_{ys}+m\left(\sqrt{u_{xs'}^2+u_{ys'}^2}\right)u_{ys},$$

where replacing the values of the table we have

$$m(w)w-m\left(\sqrt{v^2+\left(-w\sqrt{1-\frac{v^2}{c^2}}\right)^2}\right)w\sqrt{1-\frac{v^2}{c^2}}=-m(\overline{w})\overline{w}+m\left(\sqrt{v^2+\left(\overline{w}\sqrt{1-\frac{v^2}{c^2}}\right)^2}\right)\overline{w}\sqrt{1-\frac{v^2}{c^2}},$$

simplifying we have

$$m(w)=m\left(\sqrt{v^2+w^2\left(1-\frac{v^2}{c^2}\right)}\right)\sqrt{1-\frac{v^2}{c^2}}$$, where when $w\to 0$ becomes

$$m(0)=m\left(\sqrt{v^2+0^2\left(1-\frac{v^2}{c^2}\right)}\right)\sqrt{1-\frac{v^2}{c^2}}\Rightarrow m(0)=m(v)\sqrt{1-\frac{v^2}{c^2}}\Rightarrow m(v)=\frac{m(0)}{\sqrt{1-\frac{v^2}{c^2}}},$$

but $m(0)$ is equal to the resting mass m_o thus

$$m(v)=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$, with a relative velocity $v=u\Rightarrow m(u)=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$ 4.2

that applied in 4.1 supplies $\vec{p}=m(u)\vec{u}=\frac{m_0\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}$. 4.1

With the same procedures we would have for the O’ observer

$$m'(u')=\frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}$$ 4.3

and $\vec{p}'=m'(u')\vec{u}'=\frac{m_0\vec{u}'}{\sqrt{1-\frac{u'^2}{c^2}}}$. 4.1

Simplifying the simbology we will adopt $m=m(u)=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$ 4.2

and $m'=m'(u')=\frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}$ 4.3

that simplify the moments in $\vec{p}=m\vec{u}$ and $\vec{p}'=m'\vec{u}'$. 4.1

Applying 4.2 and 4.3 in 2.9 and 2.10 we have

$m=m'\sqrt{1+\frac{v^2}{c^2}+\frac{2\vec{v}\cdot\vec{u}'\vec{x}'}{c^2}}\Rightarrow m=m'\sqrt{K'}$ and $m'=m\sqrt{1+\frac{v^2}{c^2}-\frac{2\vec{v}\cdot\vec{u}\vec{x}}{c^2}}\Rightarrow m'=m\sqrt{K}$. 4.4

Defining force as Newton we have $\vec{F}=\frac{d\vec{p}}{dt}=\frac{d(m\vec{u})}{dt}$ and $\vec{F}'=\frac{d\vec{p}'}{dt}=\frac{d(m'\vec{u}')}{dt}$, with this we can define then kinetic energy (E_k, E_k) as

$$E_k=\int_0^u \vec{F}.d\vec{R}=\int_0^u \frac{d(m\vec{u})}{dt}.d\vec{R}=\int_0^u d(m\vec{u}).\vec{u}=\int_0^u (u^2dm+mudu),$$

and $E'_k=\int_0^{u'} \vec{F}'.d\vec{R}'=\int_0^{u'} \frac{d(m'\vec{u}')}{dt'}.d\vec{R}'=\int_0^{u'} d(m'\vec{u}').\vec{u}'=\int_0^{u'} (u'^2dm'+m'u'du')$.

Remodeling 4.2 and 4.3 and differentiating we have $m^2c^2-m^2u^2=m_0^2c^2\Rightarrow u^2dm+mudu=c^2dm$ and $m'^2c^2-m'^2u'^2=m_0^2c^2\Rightarrow u'^2dm'+m'u'du'=c^2dm$, that applied in the formulas of kinetic energy supplies $E_k=\int_{m_0}^{m} c^2dm=mc^2-m_0c^2=E-E_0$ and $E_k=\int_{m_0}^{m'} c^2dm'=m'c^2-m_0c^2=E'-E_0$, 4.5

where $E=mc^2$ and $E=m'c^2$ 4.6

are the total energies as in the special relativity and $E_o=m_0c^2$ 4.7
the resting energy.

Applying 4.6 in 4.4 we have exactly 2.23.

From 4.6, 4.2, 4.3, and 4.1 we find

$E=c\sqrt{m_0^2c^2+p^2}$ and $E'=c\sqrt{m_0^2c^2+p'^2}$ 4.8

identical relations to the Special Relativity.

Multiplying 2.1 and 2.2 by m_o we get

$$\frac{m_o\vec{u}'}{\sqrt{1-\frac{u'^2}{c^2}}}=\frac{m_o\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}-\frac{m_o\vec{v}}{\sqrt{1-\frac{u^2}{c^2}}}\Rightarrow m'\vec{u}'=m\vec{u}-m\vec{v}\Rightarrow\vec{p}'=\vec{p}-\frac{E}{c^2}\vec{v} \quad 4.9$$

and $$\frac{m_o\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}=\frac{m_o\vec{u}'}{\sqrt{1-\frac{u'^2}{c^2}}}-\frac{m_o\vec{v}'}{\sqrt{1-\frac{u'^2}{c^2}}}\Rightarrow m\vec{u}=m'\vec{u}'-m'\vec{v}'\Rightarrow\vec{p}=\vec{p}'-\frac{E'}{c^2}\vec{v}'. \quad 4.10$$

Table 4, transformations of moments $\vec{p}$ and $\vec{p}'$

$\vec{p}'=\vec{p}-\frac{E}{c^2}\vec{v}$	4.9	$\vec{p}=\vec{p}'-\frac{E'}{c^2}\vec{v}'$	4.10
$p'x'=px-\frac{E}{c^2}v$	4.11	$px=p'x'+\frac{E'}{c^2}v'$	4.12
$p'y'=py$	4.11.1	$py=p'y'$	4.12.1
$p'z'=pz$	4.11.2	$pz=p'z'$	4.12.2
$E'=E\sqrt{K}$	2.23	$E=E'\sqrt{K'}$	2.23
$m=m(u)=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$	4.2	$m'=m'(u')=\frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}$	4.3
$m'=m\sqrt{K}$	4.4	$m=m'\sqrt{K'}$	4.4
$E_k=E-E_o$	4.5	$E'_k=E'-E_o$	4.5
$E=mc^2$	4.6	$E'=m'c^2$	4.6
$E_o=m_oc^2$	4.7	$E_o=m_oc^2$	4.7
$E=c\sqrt{m_o^2c^2+p^2}$	4.8	$E'=c\sqrt{m_o^2c^2+p'^2}$	4.8

Wave equation of Louis de Broglie

The observer O' associates to a resting particle in its origin the following properties:

-Resting mass m_o

-Time $t'=t_o$

-Resting Energy $E_o=m_oc^2$

-Frequency $y_o=\frac{E_o}{h}=\frac{m_oc^2}{h}$

-Wave function $\psi_o=a\,sen\,2\pi y_o t_o$ with a = constant.

The observer O associates to a particle with velocity v the following:

-Mass $m=m(v)=\frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}$ (from 4.2 where $u=v$)

-Time $t=\frac{t_o}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vv}{c^2}}}=\frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}}$ (from 1.7 with $ux=v$ and $t'=t_o$)

-Energy $E=\frac{E_o}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_oc^2}{\sqrt{1-\frac{v^2}{c^2}}}$ (from 2.23 with $ux=v$ and $E'=E_o$)

-Frequency $y=\frac{y_o}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_o c^2/h}{\sqrt{1-\frac{v^2}{c^2}}}$ (from 2.22 with $u.x=v$ and $y'=y_o$)

-Distance x = vt (from 1.2 with x' = 0)

-Wave function $\psi=asen2\pi y_o t_o=asen2\pi y\sqrt{1-\frac{v^2}{c^2}}\,t\sqrt{1-\frac{v^2}{c^2}}=asen2\pi y\left(t-\frac{x}{u}\right)$ with $u=\frac{c^2}{v}$

-Wave length $u=y\lambda=\frac{c^2}{v}=\frac{E}{p}=\frac{yh}{p}\Rightarrow\lambda=\frac{h}{p}$ (from 4.9 with $\vec{p}'=\vec{p}_o=0$)

To go back to the O' observer referential where $\vec{u}'=0\Rightarrow u'x'=0$, we will consider the following variables:

-Distance x = v't' (from 1.4 with x' = 0)

-Time $t=t'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'0}{c^2}}=t'\sqrt{1+\frac{v'^2}{c^2}}$ (from 1.8 with $u'x'=0$)

-Frequency $y=y'\sqrt{1+\frac{v'^2}{c^2}}$ (from 2.22 with $u'x'=0$)

-Velocity $v=\frac{v'}{\sqrt{1+\frac{v'^2}{c^2}}}$ (de 2.13)

that applied to the wave function supplies

$$\psi'=asen2\pi y\left(t-\frac{vx}{c^2}\right)=asen2\pi y'\sqrt{1+\frac{v'^2}{c^2}}\left(t'\sqrt{1+\frac{v'^2}{c^2}}-\frac{v'^2t'}{c^2\sqrt{1+\frac{v'^2}{c^2}}}\right)=asen2\pi y't',$$

but as $t'=t_o$ and $y'=y_o$ then $\psi'=\psi_o$.

§5 Transformations of the Forces $\vec{F}$ and $\vec{F}'$

Differentiating 4.9 and dividing by 1.17 we have

$$\frac{d\vec{p}'}{dt'}=\frac{d\vec{p}}{dt\sqrt{K}}-\frac{dE}{dt\sqrt{K}}\frac{\vec{v}}{c^2}\Rightarrow\vec{F}'=\frac{1}{\sqrt{K}}\left[\vec{F}-\frac{dE}{dt}\frac{\vec{v}}{c^2}\right]\Rightarrow\vec{F}'=\frac{1}{\sqrt{K}}\left[\vec{F}-\left(\vec{F}.\vec{u}\right)\frac{\vec{v}}{c^2}\right]. \qquad 5.1$$

Differentiating 4.10 and dividing by 1.22 we have

$$\frac{d\vec{p}}{dt}=\frac{d\vec{p}'}{dt'\sqrt{K'}}-\frac{dE'}{dt'\sqrt{K'}}\frac{\vec{v}'}{c^2}\Rightarrow\vec{F}=\frac{1}{\sqrt{K'}}\left[\vec{F}'-\frac{dE'}{dt'}\frac{\vec{v}'}{c^2}\right]\Rightarrow\vec{F}=\frac{1}{\sqrt{K'}}\left[\vec{F}'-\left(\vec{F}'.\vec{u}'\right)\frac{\vec{v}'}{c^2}\right]. \qquad 5.2$$

From the system formed by 5.1 and 5.2 we have

$$\frac{dE}{dt}=\frac{dE}{dt'} \text{ or } \vec{F}.\vec{u}=\vec{F}'.\vec{u}', \qquad 5.3$$

that is an invariant between the observers in the Undulating .Relativity.

Table 5, transformations of the Forces $\vec{F}$ and $\vec{F}'$

$\vec{F}'=\frac{1}{\sqrt{K}}\left[\vec{F}-\left(\vec{F}.\vec{u}\right)\frac{\vec{v}}{c^2}\right]$	5.1	$\vec{F}=\frac{1}{\sqrt{K'}}\left[\vec{F}'-\left(\vec{F}'.\vec{u}'\right)\frac{\vec{v}'}{c^2}\right]$	5.2
$F'x'=\frac{1}{\sqrt{K}}\left[Fx-\left(\vec{F}.\vec{u}\right)\frac{v}{c^2}\right]$	5.4	$Fx=\frac{1}{\sqrt{K'}}\left[F'x'+\left(\vec{F}'.\vec{u}'\right)\frac{v'}{c^2}\right]$	5.5
$F'y'=Fy/\sqrt{K}$	5.4.1	$Fy=F'y'/\sqrt{K'}$	5.5.1
$F'z'=Fz/\sqrt{K}$	5.4.2	$Fz=F'z'/\sqrt{K'}$	5.5.2
$\frac{dE}{dt}=\frac{dE}{dt}$	5.3	$\vec{F}.\vec{u}=\vec{F}'.\vec{u}'$	5.3

§6 Transformations of the density of charge ρ, ρ' and density of current $\vec{J}$ and $\vec{J}'$

Multiplying 2.1 and 2.2 by the density of the resting electric charge defined as $\rho_o=\frac{dq}{dv_o}$ we have

$$\frac{\rho_o\vec{u}'}{\sqrt{1-\frac{u'^2}{c^2}}}=\frac{\rho_o\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}-\frac{\rho_o\vec{v}}{\sqrt{1-\frac{u^2}{c^2}}}\Rightarrow\rho'\vec{u}'=\rho\vec{u}-\rho\vec{v}\Rightarrow\vec{J}'=\vec{J}-\rho\vec{v} \quad 6.1$$

and $\frac{\rho_o\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}=\frac{\rho_o\vec{u}'}{\sqrt{1-\frac{u'^2}{c^2}}}-\frac{\rho_o\vec{v}'}{\sqrt{1-\frac{u'^2}{c^2}}}\Rightarrow\rho\vec{u}=\rho'\vec{u}'-\rho'\vec{v}'\Rightarrow\vec{J}=\vec{J}'-\rho'\vec{v}'$. 6.2

Table 6, transformations of the density of charges ρ, ρ' and density of current $\vec{J}$ and $\vec{J}'$

$\vec{J}'=\vec{J}-\rho\vec{v}$	6.1	$\vec{J}=\vec{J}'-\rho'\vec{v}'$	6.2
$J'x'=Jx-\rho v$	6.3	$Jx=J'x'+\rho'v$	6.4
$J'y'=Jy$	6.3.1	$Jy=J'y'$	6.4.1
$J'z'=Jz$	6.3.2	$Jz=J'z'$	6.4.2
$\vec{J}=\rho\vec{u}$	6.5	$\vec{J}'=\rho'\vec{u}'$	6.6
$\rho=\frac{\rho_o}{\sqrt{1-\frac{u^2}{c^2}}}$	6.7	$\rho'=\frac{\rho_o}{\sqrt{1-\frac{u'^2}{c^2}}}$	6.8
$\rho'=\rho\sqrt{K}$	6.9	$\rho=\rho'\sqrt{K'}$	6.10

From the system formed by 6.1 and 6.2 we had 6.9 and 6.10.

§7 Transformation of the electric fields $\vec{E}$, $\vec{E}'$ and magnetic fields $\vec{B}$, $\vec{B}'$

Applying the forces of Lorentz $\vec{F}=q\left(\vec{E}+\vec{u}\times\vec{B}\right)$ and $\vec{F}'=q\left(\vec{E}'+\vec{u}'\times\vec{B}'\right)$ in 5.1 and 5.2 we have

$$q\left(\vec{E}'+\vec{u}'\times\vec{B}'\right)=\frac{1}{\sqrt{K}}\left[q\left(\vec{E}+\vec{u}\times\vec{B}\right)-\left[q\left(\vec{E}+\vec{u}\times\vec{B}\right).\vec{u}\right]\frac{\vec{v}}{c^2}\right]$$

and $q\left(\vec{E}+\vec{u}\times\vec{B}\right)=\frac{1}{\sqrt{K'}}\left[q\left(\vec{E}'+\vec{u}'\times\vec{B}'\right)-\left[q\left(\vec{E}'+\vec{u}'\times\vec{B}'\right).\vec{u}'\right]\frac{\vec{v}'}{c^2}\right]$, that simplified become

$\left(\vec{E}'+\vec{u}'\times\vec{B}'\right)=\frac{1}{\sqrt{K}}\left[\left(\vec{E}+\vec{u}\times\vec{B}\right)-\left(\vec{E}.\vec{u}\right)\frac{\vec{v}}{c^2}\right]$ and $\left(\vec{E}+\vec{u}\times\vec{B}\right)=\frac{1}{\sqrt{K'}}\left[\left(\vec{E}'+\vec{u}'\times\vec{B}'\right)-\left(\vec{E}'.\vec{u}'\right)\frac{\vec{v}'}{c^2}\right]$ from where we get the invariance of $\vec{E}.\vec{u}=\vec{E}'\vec{u}'$ between the observers as a consequence of 5.3 and the following components of each axis

$$E'x'+u'y'B'z'-u'z'B'y'=\frac{1}{\sqrt{K}}\left[Ex+uyBz-uzBy-\frac{Exuxv}{c^2}-\frac{Eyuyv}{c^2}-\frac{Ezuzv}{c^2}\right] \quad 7.1$$

$$E'y'+u'z'B'x'-u'x'B'z'=\frac{1}{\sqrt{K}}\left[Ey+uzBx-uxBz\right] \quad 7.1.1$$

$$E'z'+u'x'B'y'-u'y'B'x'=\frac{1}{\sqrt{K}}\left[Ez+uxBy-uyBx\right] \quad 7.1.2$$

$$Ex+uyBz-uzBy=\frac{1}{\sqrt{K'}}\left[E'x'+u'y'B'z'-u'z'B'y'+\frac{E'x'u'x'v'}{c^2}+\frac{E'y'u'y'v'}{c^2}+\frac{E'z'u'z'v'}{c^2}\right] \quad 7.2$$

$$Ey+uzBx-uxBz=\frac{1}{\sqrt{K'}}\left[E'y'+u'z'B'x'-u'x'B'z'\right] \quad 7.2.1$$

$$Ez+uxBy-uyBx=\frac{1}{\sqrt{K'}}\left[E'z'+u'x'B'y'-u'y'B'x'\right] \quad 7.2.2$$

To the conjunct 7.1 and 7.2 we have two solutions described in the tables 7 and 8.

Table 7, transformations of the electric fields $\vec{E}$, $\vec{E}'$ and magnetic fields $\vec{B}$ e $\vec{B}'$

$E'x'=\frac{Ex}{\sqrt{K}}\left(1-\frac{vux}{c^2}\right)$	7.3	$Ex=\frac{E'x'}{\sqrt{K'}}\left(1+\frac{v'u'x'}{c^2}\right)$	7.4
$E'y'=\frac{Ey}{\sqrt{K}}\left(1+\frac{v^2}{c^2}-\frac{vux}{c^2}\right)-\frac{vBz}{\sqrt{K}}$	7.3.1	$Ey=\frac{E'y'}{\sqrt{K'}}\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'}{c^2}\right)+\frac{v'B'z'}{\sqrt{K'}}$	7.4.1
$E'z'=\frac{Ez}{\sqrt{K}}\left(1+\frac{v^2}{c^2}-\frac{vux}{c^2}\right)+\frac{vBy}{\sqrt{K}}$	7.3.2	$Ez=\frac{E'z'}{\sqrt{K'}}\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'}{c^2}\right)-\frac{v'B'y'}{\sqrt{K'}}$	7.4.2
$B'x'=Bx$	7.5	$Bx=B'x'$	7.6
$B'y'=By+\frac{v}{c^2}Ez$	7.5.1	$By=B'y'-\frac{v'}{c^2}E'z'$	7.6.1
$B'z'=Bz-\frac{v}{c^2}Ey$	7.5.2	$Bz=B'z'+\frac{v'}{c^2}E'y'$	7.6.2
$E'y'=Ey\sqrt{K}$	7.7	$Ey=E'y'\sqrt{K'}$	7.8
$E'z'=Ez\sqrt{K}$	7.7.1	$Ez=E'z'\sqrt{K'}$	7.8.1
$By=-\frac{ux}{c^2}Ez$	7.9	$B'y'=-\frac{u'x'}{c^2}E'z'$	7.10
$Bz=\frac{ux}{c^2}Ey$	7.9.1	$B'z'=\frac{u'x'}{c^2}E'y'$	7.10.1

Table 8, transformations of the electric fields $\vec{E}$, $\vec{E}'$ and magnetic fields $\vec{B}$ e $\vec{B}'$

$E'x'=\frac{1}{\sqrt{K}}\left[Ex-\left(\vec{E}.\vec{u}\right)\frac{v}{c^2}\right]$	7.11	$Ex=\frac{1}{\sqrt{K'}}\left[E'x'+\left(\vec{E}'.\vec{u}'\right)\frac{v'}{c^2}\right]$	7.12
$E'y'=\frac{1}{\sqrt{K}}\left(Ey-vBz\right)$	7.11.1	$Ey=\frac{1}{\sqrt{K'}}\left(E'y'+v'B'z'\right)$	7.12.1
$E'z'=\frac{1}{\sqrt{K}}\left(Ez+vBy\right)$	7.11.2	$Ez=\frac{1}{\sqrt{K'}}\left(E'z'-v'B'y'\right)$	7.12.2
$B'x'=Bx$	7.13	$Bx=B'x'$	7.14
$B'y'=By$	7.13.1	$By=B'y'$	7.14.1
$B'z'=Bz$	7.13.2	$Bz=B'z'$	7.14.2

Relation between the electric field and magnetic field

If an electric-magnetic field has to the observer O' the naught magnetic component $\vec{B}'=zero$ and the electric component $\vec{E}'$. To the observer O this field is represented with both components, being the magnetic field described by the conjunct 7.5 and hás as components

$$Bx=zero,\ By=-\frac{vEz}{c^2},\ Bz=\frac{vEy}{c^2}, \qquad 7.15$$

that are equivalent to $\vec{B}=\frac{1}{c^2}\vec{v}\times\vec{E}$. 7.16

Formula of Biot-Savart

The observer O' associates to a resting electric charge, uniformly distributed alongside its axis x' the following electric-magnetic properties:

-Linear density of resting electric charge $\rho_o=\frac{dq}{dx'}$

-Naught electric current $I'=zero$

-Naught magnetic field $\vec{B}'=zero\Rightarrow\vec{u}'=zero$

-Radial electrical field of module $E'=\sqrt{E'y'^2+E'z'^2}=\frac{\rho_o}{2\pi\varepsilon_o R}$ at any point of radius $R=\sqrt{y'^2+z'^2}$ with the component $E'x'=zero$.

To the observer O it relates to an electric charge uniformly distributed alongside its axis with velocity $ux=v$ to which it associates the following electric-magnetic properties:

-Linear density of the electric charge $\rho=\frac{\rho_o}{\sqrt{1-\frac{v^2}{c^2}}}$ (from 6.7 with u = v)

-Electric current $I=\rho v=\frac{\rho_o v}{\sqrt{1-\frac{v^2}{c^2}}}$

-Radial electrical field of module $E=\frac{E'}{\sqrt{1-\frac{v^2}{c^2}}}$ (according to the conjuncts 7.3 and 7.5 with $\vec{B}'=zero\Rightarrow\vec{u}'=zero$ and $ux=v$)

-Magnetic field of components $Bx=zero$, $By=-\frac{vEz}{c^2}$, $Bz=\frac{vEy}{c^2}$ and module $B=\frac{vE}{c^2}=\frac{v}{c^2}\frac{E'}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{v}{c^2}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\rho_o}{2\pi\varepsilon_o R}=\frac{\mu_o I}{2\pi R}$ where $\mu_o=\frac{1}{\varepsilon_o c^2}$, being in the vectorial form

$$\vec{B}=\frac{\mu_o I}{2\pi R}\vec{u} \qquad 7.17$$

where $\vec{u}$ is a unitary vector perpendicular to the electrical field $\vec{E}$ and tangent to the circumference that passes by the point of radius $R=\sqrt{y^2+z^2}$ because from the conjunct 7.4 and 7.6 $\vec{E}.\vec{B}=zero$.

§8 Transformations of the differential operators

Table 9, differential operators

$\frac{\partial}{\partial x'}=\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}$	8.1	$\frac{\partial}{\partial x}=\frac{\partial}{\partial x'}-\frac{v'}{c^2}\frac{\partial}{\partial t'}$	8.2
$\frac{\partial}{\partial y'}=\frac{\partial}{\partial y}$	8.1.1	$\frac{\partial}{\partial y}=\frac{\partial}{\partial y'}$	8.2.1
$\frac{\partial}{\partial z'}=\frac{\partial}{\partial z}$	8.1.2	$\frac{\partial}{\partial z}=\frac{\partial}{\partial z'}$	8.2.2
$\frac{\partial}{\partial t'}=\frac{v}{\sqrt{K}}\frac{\partial}{\partial x}+\frac{1}{\sqrt{K}}\left(1+\frac{v^2}{c^2}-\frac{vx}{c^2 t}\right)\frac{\partial}{\partial t}$	8.3	$\frac{\partial}{\partial t}=-\frac{v'}{\sqrt{K'}}\frac{\partial}{\partial x'}+\frac{1}{\sqrt{K'}}\left(1+\frac{v'^2}{c^2}+\frac{v'x'}{c^2 t'}\right)\frac{\partial}{\partial t'}$	89.4

From the system formed by 8.1, 8.2, 8.3, and 8.4 and with 1.15 and 1.20 we only find the solutions

$$\frac{\partial}{\partial x}+\frac{x/t}{c^2}\frac{\partial}{\partial t}=o \text{ and } \frac{\partial}{\partial x'}+\frac{x'/t'}{c^2}\frac{\partial}{\partial t'}=o. \quad 8.5$$

From where we conclude that only the functions ψ (2.19) and ψ' (2.20) that supply the conditions

$$\frac{\partial\psi}{\partial x}+\frac{x/t}{c^2}\frac{\partial\psi}{\partial t}=o \text{ and } \frac{\partial\psi'}{\partial x'}+\frac{x'/t'}{c^2}\frac{\partial\psi'}{\partial t'}=o, \quad 8.6$$

can represent the propagation with velocity c in the Undulating Relativity indicating that the field propagates with definite velocity and without distortion being applied to 1.13 and 1.18. Because of symmetry we can also write to the other axis

$$\frac{\partial\psi}{\partial y}+\frac{y/t}{c^2}\frac{\partial\psi}{\partial t}=o,\ \frac{\partial\psi'}{\partial y'}+\frac{y'/t'}{c^2}\frac{\partial\psi'}{\partial t'}=o \text{ and } \frac{\partial\psi}{\partial z}+\frac{z/t}{c^2}\frac{\partial\psi}{\partial t}=o,\ \frac{\partial\psi'}{\partial z'}+\frac{z'/t'}{c^2}\frac{\partial\psi'}{\partial t'}=o. \quad 8.7$$

From the transformations of space and time of the Undulatory Relativity we get to Jacob's theorem

$$J=\frac{\partial(x',y',z',t')}{\partial(x,y,z,t)}=\frac{1-\frac{vux}{c^2}}{\sqrt{K}} \text{ and } J'=\frac{\partial(x,y,z,t)}{\partial(x',y',z',t')}=\frac{1+\frac{v'u'x'}{c^2}}{\sqrt{K'}}, \quad 8.8$$

variables with ux and u'x' as a consequence of the principle of contancy of the light velocity but are equal ais $J=J'$ and will be equal to one $J=J'=1$ when $ux=u'x'=c$.

Invariance of the wave equation

The wave equation to the observer O' is

$$\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}=zero$$

where applying to the formulas of tables 9 and 1.13 we get

$$\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)^2+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\left[\frac{v}{\sqrt{K}}\frac{\partial}{\partial x}+\frac{1}{\sqrt{K}}\left(1+\frac{v^2}{c^2}-\frac{vux}{c^2}\right)\frac{\partial}{\partial t}\right]^2=zero$$

from where we find

$$K\frac{\partial^2}{\partial x^2}+K\frac{\partial^2}{\partial y^2}+K\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{2v}{c^2}\frac{\partial^2}{\partial x\partial t}+\frac{2v^3}{c^4}\frac{\partial^2}{\partial x\partial t}-\frac{4v^2ux}{c^4}\frac{\partial^2}{\partial x\partial t}+\frac{v^2}{c^4}\frac{\partial^2}{\partial t^2}+\frac{v^4}{c^6}\frac{\partial^2}{\partial t^2}-\frac{2v^3ux}{c^6}\frac{\partial^2}{\partial t^2}-$$
$$-\frac{v^2}{c^2}\frac{\partial^2}{\partial x^2}-\frac{2v}{c^2}\frac{\partial^2}{\partial x\partial t}-\frac{2v^3}{c^4}\frac{\partial^2}{\partial x\partial t}+\frac{2v^2ux}{c^4}\frac{\partial^2}{\partial x\partial t}-\frac{2v^2}{c^4}\frac{\partial^2}{\partial t^2}+\frac{2vux}{c^4}\frac{\partial^2}{\partial t^2}+\frac{2v^3ux}{c^6}\frac{\partial^2}{\partial t^2}-\frac{v^2ux^2}{c^6}\frac{\partial^2}{\partial t^2}-\frac{v^4}{c^6}\frac{\partial^2}{\partial t^2}=zero$$

that simplifying supplies
$$K\frac{\partial^2}{\partial x^2}+K\frac{\partial^2}{\partial y^2}+K\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{2v^2ux}{c^4}\frac{\partial^2}{\partial x\partial t}-\frac{v^2}{c^2}\frac{\partial^2}{\partial x^2}-\frac{v^2}{c^4}\frac{\partial^2}{\partial t^2}+\frac{2vux}{c^4}\frac{\partial^2}{\partial t^2}-\frac{v^2ux^2}{c^6}\frac{\partial^2}{\partial t^2}=zero$$

where reordering the terms we find
$$K\frac{\partial^2}{\partial x^2}+K\frac{\partial^2}{\partial y^2}+K\frac{\partial^2}{\partial z^2}-\left(1+\frac{v^2}{c^2}-\frac{2vux}{c^2}\right)\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{v^2}{c^2}\left(\frac{\partial^2}{\partial x^2}+\frac{2ux}{c^2}\frac{\partial^2}{\partial x\partial t}+\frac{ux^2}{c^4}\frac{\partial^2}{\partial t^2}\right)=zero \qquad 8.9$$

but from 8.5 and 1.13 we have
$$\frac{\partial}{\partial x}+\frac{x/t}{c^2}\frac{\partial}{\partial t}=o\Rightarrow\left(\frac{\partial}{\partial x}+\frac{ux}{c^2}\frac{\partial}{\partial t}\right)^2=\frac{\partial^2}{\partial x^2}+\frac{2ux}{c^2}\frac{\partial^2}{\partial x\partial t}+\frac{ux^2}{c^4}\frac{\partial^2}{\partial t^2}=zero$$

that applied in 8.9 supplies the wave equation to the observer O $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}=zero.$ 8.10

To return to the referential of the observer O' we will apply 8.10 to the formulas of tables 9 and 1.18, getting
$$\left(\frac{\partial}{\partial x'}-\frac{v'}{c^2}\frac{\partial}{\partial t'}\right)^2+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\left[-\frac{v'}{\sqrt{K'}}\frac{\partial}{\partial x'}+\frac{1}{\sqrt{K'}}\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'}{c^2}\right)\frac{\partial}{\partial t'}\right]^2=zero$$

from where we find
$$K'\frac{\partial^2}{\partial x'^2}+K'\frac{\partial^2}{\partial y'^2}+K'\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}-\frac{2v'}{c^2}\frac{\partial^2}{\partial x'\partial t'}-\frac{2v'^3}{c^4}\frac{\partial^2}{\partial x'\partial t'}-\frac{4v'^2u'x'}{c^4}\frac{\partial^2}{\partial x'\partial t'}+\frac{v'^2}{c^4}\frac{\partial^2}{\partial t'^2}+\frac{v'^4}{c^6}\frac{\partial^2}{\partial t'^2}+$$
$$+\frac{2v'^3u'x'}{c^6}\frac{\partial^2}{\partial t'^2}-\frac{v'^2}{c^2}\frac{\partial^2}{\partial x'^2}+\frac{2v'}{c^2}\frac{\partial^2}{\partial x'\partial t'}+\frac{2v'^3}{c^4}\frac{\partial^2}{\partial x'\partial t'}+\frac{2v'^2u'x'}{c^4}\frac{\partial^2}{\partial x'\partial t'}-\frac{2v'^2}{c^4}\frac{\partial^2}{\partial t'^2}-\frac{2v'u'x'}{c^4}\frac{\partial^2}{\partial t'^2}-$$
$$-\frac{v'^3u'x'}{c^6}\frac{\partial^2}{\partial t'^2}-\frac{v'^2u'x'^2}{c^6}\frac{\partial^2}{\partial t'^2}-\frac{v'^4}{c^6}\frac{\partial^2}{\partial t'^2}=zero$$

that simplifying supplies
$$K'\frac{\partial^2}{\partial x'^2}+K'\frac{\partial^2}{\partial y'^2}+K'\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}-\frac{2v'^2u'x'}{c^4}\frac{\partial^2}{\partial x'\partial t'}-\frac{v'^2}{c^2}\frac{\partial^2}{\partial x'^2}-\frac{v'^2}{c^4}\frac{\partial^2}{\partial t'^2}-\frac{2v'u'x'}{c^4}\frac{\partial^2}{\partial t'^2}-\frac{v'^2u'x'^2}{c^6}\frac{\partial^2}{\partial t'^2}=zero$$

where reordering the terms we find
$$K'\frac{\partial^2}{\partial x'^2}+K'\frac{\partial^2}{\partial y'^2}+K'\frac{\partial^2}{\partial z'^2}-\left(1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}\right)\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}-\frac{v'^2}{c^2}\left(\frac{\partial^2}{\partial x'^2}+\frac{2u'x'}{c^2}\frac{\partial^2}{\partial x'\partial t'}+\frac{u'x'^2}{c^4}\frac{\partial^2}{\partial t'^2}\right)=zero$$

but from 8.5 and 1.18 we have
$$\frac{\partial}{\partial x'}+\frac{x'/t'}{c^2}\frac{\partial}{\partial t'}=o\Rightarrow\left(\frac{\partial}{\partial x'}+\frac{u'x'}{c^2}\frac{\partial}{\partial t'}\right)^2=\frac{\partial^2}{\partial x'^2}+\frac{2u'x'}{c^2}\frac{\partial^2}{\partial x'\partial t'}+\frac{u'x'^2}{c^4}\frac{\partial^2}{\partial t'^2}=zero$$

that replaced in the reordered equation supplies the wave equation to the observer O'.

Invariance of the Continuity equation

The continuity equation in the differential form to the observer O' is

$$\frac{\partial\rho'}{\partial t'}+\nabla.J'=zero\Rightarrow\frac{\partial\rho'}{\partial t'}+\frac{\partial Jx'}{\partial x'}+\frac{\partial Jy'}{\partial y'}+\frac{\partial Jz'}{\partial z'}=zero \qquad 8.11$$

where replacing the formulas of tables 6, 9, and 1.13 we get
$$\left(\frac{v}{\sqrt{K}}\frac{\partial}{\partial x}+\frac{1}{\sqrt{K}}\left(1+\frac{v^2}{c^2}-\frac{vux}{c^2}\right)\frac{\partial}{\partial t}\right)\rho\sqrt{K}+\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)(Jx-\rho v)+\frac{\partial Jy}{\partial y}+\frac{\partial Jz}{\partial z}=zero$$

making the operations we find

$$\frac{v\partial\rho}{\partial x}+\frac{\partial\rho}{\partial t}+\frac{v^2}{c^2}\frac{\partial\rho}{\partial t}-\frac{vux}{c^2}\frac{\partial\rho}{\partial t}+\frac{\partial Jx}{\partial x}+\frac{v}{c^2}\frac{\partial Jx}{\partial t}-\frac{v\partial\rho}{\partial x}-\frac{v^2}{c^2}\frac{\partial\rho}{\partial t}+\frac{\partial Jy}{\partial y}+\frac{\partial Jz}{\partial z}=zero$$

that simplifying supplies

$$\frac{\partial\rho}{\partial t}-\frac{vux}{c^2}\frac{\partial\rho}{\partial t}+\frac{\partial Jx}{\partial x}+\frac{v}{c^2}\frac{\partial Jx}{\partial t}+\frac{\partial Jy}{\partial y}+\frac{\partial Jz}{\partial z}=zero$$

where applying $Jx=\rho ux$ with ux constant we get

$$\frac{\partial\rho}{\partial t}-\frac{vux}{c^2}\frac{\partial\rho}{\partial t}+\frac{\partial Jx}{\partial x}+\frac{v}{c^2}\frac{\partial(\rho ux)}{\partial t}+\frac{\partial Jy}{\partial y}+\frac{\partial Jz}{\partial z}=zero\Rightarrow\frac{\partial\rho}{\partial t}+\frac{\partial Jx}{\partial x}+\frac{\partial Jy}{\partial y}+\frac{\partial Jz}{\partial z}=zero \qquad 8.12$$

that is the continuity equation in the differential form to the observer O.

To get again the continuity equation in the differential form to the observer O' we will replace the formulas of tables 6, 9, and 1.18 in 8.12 getting

$$\left(-\frac{v'}{\sqrt{K'}}\frac{\partial}{\partial x'}+\frac{1}{\sqrt{K'}}\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'}{c^2}\right)\frac{\partial}{\partial t'}\right)\rho'\sqrt{K'}+\left(\frac{\partial}{\partial x'}-\frac{v'}{c^2}\frac{\partial}{\partial t'}\right)(J'x'+\rho'v')+\frac{\partial J'y'}{\partial y'}+\frac{\partial J'z'}{\partial z'}=zero$$

making the operations we find

$$-\frac{v'\partial\rho'}{\partial x'}+\frac{\partial\rho'}{\partial t'}+\frac{v'^2}{c^2}\frac{\partial\rho'}{\partial t'}+\frac{v'u'x'}{c^2}\frac{\partial\rho'}{\partial t'}+\frac{\partial J'x'}{\partial x'}-\frac{v'}{c^2}\frac{\partial J'x'}{\partial t'}+\frac{v'\partial\rho'}{\partial x'}-\frac{v'^2}{c^2}\frac{\partial\rho'}{\partial t'}+\frac{\partial J'y'}{\partial y'}+\frac{\partial J'z'}{\partial z'}=zero$$

that simplifying supplies

$$\frac{\partial\rho'}{\partial t'}+\frac{v'u'x'}{c^2}\frac{\partial\rho'}{\partial t'}+\frac{\partial J'x'}{\partial x'}-\frac{v'}{c^2}\frac{\partial J'x'}{\partial t'}+\frac{\partial J'y'}{\partial y'}+\frac{\partial J'z'}{\partial z'}=zero$$

where applying $J'x'=\rho'u'x'$ with u'x' constant we get

$$\frac{\partial\rho'}{\partial t'}+\frac{v'u'x'}{c^2}\frac{\partial\rho'}{\partial t'}+\frac{\partial J'x'}{\partial x'}-\frac{v'}{c^2}\frac{\partial(\rho'u'x')}{\partial t'}+\frac{\partial J'y'}{\partial y'}+\frac{\partial J'z'}{\partial z'}=zero\Rightarrow\frac{\partial\rho'}{\partial t'}+\frac{\partial J'x'}{\partial x'}+\frac{\partial J'y'}{\partial y'}+\frac{\partial J'z'}{\partial z'}=zero$$

that is the continuity equation in the differential form to the observer O'.

Invariance of Maxwell's equations

That in the differential form are written this way

With electrical charge

To the observer O		To the observer O'	
$\frac{\partial Ex}{\partial x}+\frac{\partial Ey}{\partial y}+\frac{\partial Ez}{\partial z}=\frac{\rho}{\varepsilon_o}$	8.13	$\frac{\partial E'x'}{\partial x'}+\frac{\partial E'y'}{\partial y'}+\frac{\partial E'z'}{\partial z'}=\frac{\rho'}{\varepsilon_o}$	8.14
$\frac{\partial Bx}{\partial x}+\frac{\partial By}{\partial y}+\frac{\partial Bz}{\partial z}=0$	8.15	$\frac{\partial B'x'}{\partial x'}+\frac{\partial B'y'}{\partial y'}+\frac{\partial B'z'}{\partial z'}=0$	8.16
$\frac{\partial Ey}{\partial x}-\frac{\partial Ex}{\partial y}=-\frac{\partial Bz}{\partial t}$	8.17	$\frac{\partial E'y'}{\partial x'}-\frac{\partial E'x'}{\partial y'}=-\frac{\partial B'z'}{\partial t'}$	8.18
$\frac{\partial Ez}{\partial y}-\frac{\partial Ey}{\partial z}=-\frac{\partial Bx}{\partial t}$	8.19	$\frac{\partial E'z'}{\partial y'}-\frac{\partial E'y'}{\partial z'}=-\frac{\partial B'x'}{\partial t'}$	8.20
$\frac{\partial Ex}{\partial z}-\frac{\partial Ez}{\partial x}=-\frac{\partial By}{\partial t}$	8.21	$\frac{\partial E'x'}{\partial z'}-\frac{\partial E'z'}{\partial x'}=-\frac{\partial B'y'}{\partial t'}$	8.22
$\frac{\partial By}{\partial x}-\frac{\partial Bx}{\partial y}=\mu_o Jz+\varepsilon_o\mu_o\frac{\partial Ez}{\partial t}$	8.23	$\frac{\partial B'y'}{\partial x'}-\frac{\partial B'x'}{\partial y'}=\mu_o J'z'+\varepsilon_o\mu_o\frac{\partial E'z'}{\partial t'}$	8.24

$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\mu_o Jx+\varepsilon_o\mu_o\frac{\partial Ex}{\partial t}$	8.25	$\frac{\partial B'z'}{\partial y'}-\frac{\partial B'y'}{\partial z'}=\mu_o J'x'+\varepsilon_o\mu_o\frac{\partial E'x'}{\partial t'}$	8.26
$\frac{\partial Bx}{\partial z}-\frac{\partial Bz}{\partial x}=\mu_o Jy+\varepsilon_o\mu_o\frac{\partial Ey}{\partial t}$	8.27	$\frac{\partial B'x'}{\partial z'}-\frac{\partial B'z'}{\partial x'}=\mu_o J'y'+\varepsilon_o\mu_o\frac{\partial E'y'}{\partial t'}$	8.28

Without electrical charge $\rho=\rho'=zero$ and $J=J'=zero$

To the observer O		To the observer O’	
$\frac{\partial Ex}{\partial x}+\frac{\partial Ey}{\partial y}+\frac{\partial Ez}{\partial z}=0$	8.29	$\frac{\partial E'x'}{\partial x'}+\frac{\partial E'y'}{\partial y'}+\frac{\partial E'z'}{\partial z'}=0$	8.30
$\frac{\partial Bx}{\partial x}+\frac{\partial By}{\partial y}+\frac{\partial Bz}{\partial z}=0$	8.31	$\frac{\partial B'x'}{\partial x'}+\frac{\partial B'y'}{\partial y'}+\frac{\partial B'z'}{\partial z'}=0$	8.32
$\frac{\partial Ey}{\partial x}-\frac{\partial Ex}{\partial y}=-\frac{\partial Bz}{\partial t}$	8.33	$\frac{\partial E'y'}{\partial x'}-\frac{\partial E'x'}{\partial y'}=-\frac{\partial B'z'}{\partial t'}$	8.34
$\frac{\partial Ez}{\partial y}-\frac{\partial Ey}{\partial z}=-\frac{\partial Bx}{\partial t}$	8.35	$\frac{\partial E'z'}{\partial y'}-\frac{\partial E'y'}{\partial z'}=-\frac{\partial B'x'}{\partial t'}$	8.36
$\frac{\partial Ex}{\partial z}-\frac{\partial Ez}{\partial x}=-\frac{\partial By}{\partial t}$	8.37	$\frac{\partial E'x'}{\partial z'}-\frac{\partial E'z'}{\partial x'}=-\frac{\partial B'y'}{\partial t'}$	8.38
$\frac{\partial By}{\partial x}-\frac{\partial Bx}{\partial y}=\varepsilon_o\mu_o\frac{\partial Ez}{\partial t}$	8.39	$\frac{\partial B'y'}{\partial x'}-\frac{\partial B'x'}{\partial y'}=\varepsilon_o\mu_o\frac{\partial E'z'}{\partial t'}$	8.40
$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\varepsilon_o\mu_o\frac{\partial Ex}{\partial t}$	8.41	$\frac{\partial B'z'}{\partial y'}-\frac{\partial B'y'}{\partial z'}=\varepsilon_o\mu_o\frac{\partial E'x'}{\partial t'}$	8.42
$\frac{\partial Bx}{\partial z}-\frac{\partial Bz}{\partial x}=\varepsilon_o\mu_o\frac{\partial Ey}{\partial t}$	8.43	$\frac{\partial B'x'}{\partial z'}-\frac{\partial B'z'}{\partial x'}=\varepsilon_o\mu_o\frac{\partial E'y'}{\partial t'}$	8.44
$\varepsilon_o\mu_o=\frac{1}{c^2}$	8.45		

We demonstrate the invariance of the Law of Gauss in the differential form that for the observer O’ is

$$\frac{\partial E'x'}{\partial x'}+\frac{\partial E'y'}{\partial y'}+\frac{\partial E'z'}{\partial z'}=\frac{\rho'}{\varepsilon_o} \qquad 8.14$$

where replacing the formulas from the tables 6, 7, 9, and 1.18, and considering u’x’ constant, we get

$$\left[\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right]\frac{Ex}{\sqrt{K}}\left(1-\frac{vux}{c^2}\right)+\frac{\partial}{\partial y}\left[\frac{Ey}{\sqrt{K}}\left(1+\frac{v^2}{c^2}-\frac{vux}{c^2}\right)-\frac{vBz}{\sqrt{K}}\right]+$$

$$+\frac{\partial}{\partial z}\left[\frac{Ez}{\sqrt{K}}\left(1+\frac{v^2}{c^2}-\frac{vux}{c^2}\right)+\frac{vBy}{\sqrt{K}}\right]=\frac{\rho\sqrt{K}}{\varepsilon_o}$$

making the products, summing and subtracting the term $\frac{v^2}{c^2}\frac{\partial Ex}{\partial x}$, we find

$$\frac{\partial Ex}{\partial x}+\frac{v}{c^2}\frac{\partial Ex}{\partial t}-\frac{vux}{c^2}\frac{\partial Ex}{\partial x}-\frac{v^2ux}{c^4}\frac{\partial Ex}{\partial t}+\frac{\partial Ey}{\partial y}+\frac{v^2}{c^2}\frac{\partial Ey}{\partial y}-\frac{vux}{c^2}\frac{\partial Ey}{\partial y}-\frac{v\partial Bz}{\partial y}+$$

$$+\frac{\partial Ez}{\partial z}+\frac{v^2}{c^2}\frac{\partial Ez}{\partial z}-\frac{vux}{c^2}\frac{\partial Ez}{\partial z}+\frac{v\partial By}{\partial z}+\frac{v^2}{c^2}\frac{\partial Ex}{\partial x}-\frac{v^2}{c^2}\frac{\partial Ex}{\partial x}=\frac{\rho K}{\varepsilon_o}$$

that reordering results

$$-\frac{v^2}{c^2}\left(\frac{\partial Ex}{\partial x}+\frac{ux}{c^2}\frac{\partial Ex}{\partial t}\right)-v\left(\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}-\frac{1}{c^2}\frac{\partial Ex}{\partial t}\right)+\left(\frac{\partial Ex}{\partial x}+\frac{\partial Ey}{\partial y}+\frac{\partial Ez}{\partial z}\right)\left(1+\frac{v^2}{c^2}-\frac{vux}{c^2}\right)=\frac{\rho K}{\varepsilon_o}$$

where the first parentheses is 8.5 and because of this equal to zero , the second blank is equal to $-v(\mu_o Jx)=-v\mu_o\rho ux=-\frac{v\rho ux}{\varepsilon_o c^2}$ gotten from 8.25 and 8.45 resulting in

$$\left(\frac{\partial Ex}{\partial x}+\frac{\partial Ey}{\partial y}+\frac{\partial Ez}{\partial z}\right)\left(1+\frac{v^2}{c^2}-\frac{vux}{c^2}\right)=\frac{\rho}{\varepsilon_o}\left(1+\frac{v^2}{c^2}-\frac{vux}{c^2}\right)-\frac{\rho}{\varepsilon_o}\frac{vux}{c^2}+\frac{\rho}{\varepsilon_o}\frac{vux}{c^2}$$

from where we get $\frac{\partial Ex}{\partial x}+\frac{\partial Ey}{\partial y}+\frac{\partial Ez}{\partial z}=\frac{\rho}{\varepsilon_o}$ 8.13

that is the Law of Gauss in the differential form to the observer O.

To make the inverse we will replace in 8.13 the formulas of the tables 6, 7, 9, and 1.13, and considering ux constant, we get

$$\left[\frac{\partial}{\partial x'}-\frac{v'}{c^2}\frac{\partial}{\partial t'}\right]\frac{Ex'}{\sqrt{K'}}\left(1+\frac{v'u'x'}{c^2}\right)+\frac{\partial}{\partial y'}\left[\frac{Ey'}{\sqrt{K'}}\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'}{c^2}\right)+\frac{v'Bz'}{\sqrt{K'}}\right]+$$

$$+\frac{\partial}{\partial z'}\left[\frac{Ez'}{\sqrt{K'}}\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'}{c^2}\right)-\frac{v'By'}{\sqrt{K'}}\right]=\frac{\rho'\sqrt{K'}}{\varepsilon_o}$$

making the products, adding and subtracting the term $\frac{v'^2}{c^2}\frac{\partial Ex'}{\partial x'}$, we get

$$\frac{\partial Ex'}{\partial x'}-\frac{v'}{c^2}\frac{\partial Ex'}{\partial t'}+\frac{v'u'x'}{c^2}\frac{\partial Ex'}{\partial x'}-\frac{v'^2u'x'}{c^4}\frac{\partial Ex'}{\partial t'}+\frac{\partial Ey'}{\partial y'}+\frac{v'^2}{c^2}\frac{\partial Ey'}{\partial y'}+\frac{v'u'x'}{c^2}\frac{\partial Ey'}{\partial y'}+$$

$$+\frac{v'\partial B'z'}{\partial y'}+\frac{\partial Ez'}{\partial z'}+\frac{v'^2}{c^2}\frac{\partial Ez'}{\partial z'}+\frac{v'u'x'}{c^2}\frac{\partial Ez'}{\partial z'}-\frac{v'\partial B'y'}{\partial z'}+\frac{v'^2}{c^2}\frac{\partial Ex'}{\partial x'}-\frac{v'^2}{c^2}\frac{\partial Ex'}{\partial x'}=\frac{\rho' K'}{\varepsilon_o}$$

that reordering results in

$$-\frac{v'^2}{c^2}\left(\frac{\partial Ex'}{\partial x'}+\frac{u'x'}{c^2}\frac{\partial Ex'}{\partial t'}\right)+v'\left(\frac{\partial B'z'}{\partial y'}-\frac{\partial B'y'}{\partial z'}-\frac{1}{c^2}\frac{\partial Ex'}{\partial t'}\right)+$$

$$+\left(\frac{\partial Ex'}{\partial x'}+\frac{\partial Ey'}{\partial y'}+\frac{\partial Ez''}{\partial z'}\right)\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'}{c^2}\right)=\frac{\rho' K'}{\varepsilon_o}$$

where the first blank is 8.5 and because of this equals to zero, the second blank is equal to $v'(\mu_o J'x')=v'\mu_o\rho' u'x'=\frac{v'\rho' u'x'}{\varepsilon_o c^2}$ gotten from 8.26 and 8.45 resulting in

$$\left(\frac{\partial Ex'}{\partial x'}+\frac{\partial Ey'}{\partial y'}+\frac{\partial Ez'}{\partial z'}\right)\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'}{c^2}\right)=\frac{\rho'}{\varepsilon_o}\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'}{c^2}\right)+\frac{\rho}{\varepsilon_o}\frac{v'u'x'}{c^2}-\frac{\rho'}{\varepsilon_o}\frac{v'u'x'}{c^2}$$

from where we get $\frac{\partial E'x'}{\partial x'}+\frac{\partial Ey'}{\partial y'}+\frac{\partial Ez'}{\partial z'}=\frac{\rho'}{\varepsilon_o}$ that is the Law of Gauss in the differential form to the O' observer.

Proceeding this way we can prove the invariance of form for all the other equations of Maxwell.

§9 Explaining the Sagnac Effect with the Undulating Relativity

We must transform the straight movement of the two observers O and O' used in the deduction of the Undulating Relativity in a plain circular movement with a constant radius. Let's imagine that the observer O sees the observer O' turning around with a tangential speed v in a clockwise way (C) equals to the positive course of the axis x of UR and that the observer O' sees the observer O turning around with a tangecial speed v' in a unclockwise way (U) equals to the negative course of the axis x of the UR.

In the moment t = t' = zero, the observer O emits two rays of light from the common origin to both observers, one in a unclockwise way of arc ct_U and another in a clockwise way of arc ct_C, therefore $ct_U = ct_C$ and $t_U = t_C$, because c is the speed of the constant light, and t_U and t_C the time.

In the moment t = t' = zero the observer O' also emits two rays of light from the common origin to both observers, one in a unclockwise way (useless) of arc ct'_U and another one in a clockwise way of arc ct'_C, thus $ct'_U = ct'_C$ and $t'_U = t'_C$ because c is the speed of the constant light, and t'_U and t'_C the time.

Rewriting the equations 1.15 and 1.20 of the Undulating Relativity (UR):

$$\frac{|v'|}{|v|} = \frac{t'}{t} = \sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}. \qquad 1.15$$

$$\frac{|v|}{|v'|} = \frac{t}{t'} = \sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}. \qquad 1.20$$

Making ux = u'x' = c (ray of light projected alongside the positive axis x) and splitting the equations we have:

$$t' = t\left(1-\frac{v}{c}\right) \quad 9.1 \qquad t = t'\left(1+\frac{v'}{c}\right) \quad 9.2$$

$$v' = \frac{v}{\left(1-\frac{v}{c}\right)} \quad 9.3 \qquad v = \frac{v'}{\left(1+\frac{v'}{c}\right)} \quad 9.4$$

When the origin of the observer O' detects the unclockwise ray of the observer O, will be at the distance $vt_C = v't'_U$ of the observer O and simultaneously will detect its clockwise ray of light at the same point of the observer O, in a symmetric position to the diameter that goes through the observer O because $ct_U = ct_C \Rightarrow t_U = t_C$ and $ct'_U = ct'_C \Rightarrow t'_U = t'_C$, following the four equations above we have:

$$ct_U + vt_C = 2\pi R \Rightarrow t_C = \frac{2\pi R}{c+v} \qquad 9.5$$

$$ct'_C + 2v't'_U = 2\pi R \Rightarrow t'_C = \frac{2\pi R}{c+2v'} \qquad 9.6$$

When the origin of the observer O' detects the clockwise ray of the observer O, simultaneously will detect its own clockwise ray and will be at the distance $vt_{2C} = v't'_{2U}$ of the observer O, then following the equations 1,2,3 and 4 above we have:

$$ct_{2C} = 2\pi R + vt_{2C} \Rightarrow t_{2C} = \frac{2\pi R}{c-v} \qquad 9.7$$

$$ct'_{2C} = 2\pi R \Rightarrow t'_{2C} = \frac{2\pi R}{c} \qquad 9.8$$

The time difference to the observer O is:

$$\Delta t = t_{2C} - t_C = \frac{2\pi R}{c-v} - \frac{2\pi R}{c+v} = \frac{4\pi R v}{c^2 - v^2} \qquad 9.9$$

The time difference to the observer O' is:

$$\Delta t' = t'_{2C} - t'_C = \frac{2\pi R}{c} - \frac{2\pi R}{c+2v'} = \frac{4\pi R v'}{(c+2v')c} \qquad 9.10$$

Replacing the equations 5 to 10 in 1 to 4 we prove that they confirm the transformations of the Undulating Relativity.

§10 Explaining the experience of Ives-Stilwell with the Undulating Relativity

We should rewrite the equations (2.21) to the wave length in the Undulating Relativity:

$$\lambda' = \frac{\lambda}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}} \text{ and } \lambda = \frac{\lambda'}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}}, \qquad 2.21$$

Making ux = u'x' = c (Ray of light projected alongside the positive axis x), we have the equations:

$$\lambda' = \frac{\lambda}{\left(1-\frac{v}{c}\right)} \text{ and } \lambda = \frac{\lambda'}{\left(1+\frac{v'}{c}\right)}, \qquad 10.1$$

If the observer O, who sees the observer O' going away with the velocity v in the positive way of the axis x, emits waves, provenient of a resting source in its origin with velocity c and wave length λ_F in the positive way of the axis x, then according to the equation 10.1 the observer O' will measure the waves with velocity c and the wave length λ'_D according to the formulas:

$$\lambda'_D = \frac{\lambda_F}{\left(1-\frac{v}{c}\right)} \text{ and } \lambda_F = \frac{\lambda'_D}{\left(1+\frac{v'}{c}\right)}, \qquad 10.2$$

If the observer O', who sees the obsesrver O going away with velocity v' in the negative way of the axis x, emits waves, provenient of a resting source in its origin with velocity c and the wave length λ'_F in the positive way of the axis x, then according to the equation 10.1 the observer O will measure waves with velocity c and wave lenght λ_A according to the formulas:

$$\lambda'_F = \frac{\lambda_A}{\left(1-\frac{v}{c}\right)} \text{ and } \lambda_A = \frac{\lambda'_F}{\left(1+\frac{v'}{c}\right)}, \qquad 10.3$$

The resting sources in the origin of the observers O and O' are identical thus $\lambda_F = \lambda'_F$.

We calculate the average wave length $\bar{\lambda}$ of the measured waves (λ_A, λ'_D) using the equations 10.2 and 10.3, the left side in each equation:

$$\bar{\lambda} = \frac{\lambda'_D+\lambda_A}{2} = \frac{1}{2}\left[\frac{\lambda_F}{\left(1-\frac{v}{c}\right)} + \lambda'_F\left(1-\frac{v}{c}\right)\right] \Rightarrow \bar{\lambda} = \frac{\lambda'_D+\lambda_A}{2} = \frac{\lambda_F}{2\left(1-\frac{v}{c}\right)}\left[1+\left(1-\frac{v}{c}\right)^2\right]$$

We calculate the diffrence between the average wave length $\bar{\lambda}$ and the emited wave length by the sources $\Delta\bar{\lambda} = \bar{\lambda} - \lambda_F$:

$$\Delta\bar{\lambda} = \bar{\lambda} - \lambda_F = \frac{\lambda_F}{2\left(1-\frac{v}{c}\right)}\left[1+\left(1-\frac{v}{c}\right)^2\right] - \lambda_F$$

$$\Delta\bar{\lambda} = \frac{\lambda_F}{2\left(1-\frac{v}{c}\right)}\left[1+\left(1-\frac{v}{c}\right)^2\right] - \lambda_F\frac{2\left(1-\frac{v}{c}\right)}{2\left(1-\frac{v}{c}\right)}$$

$$\Delta\bar{\lambda} = \frac{\lambda_F}{2\left(1-\frac{v}{c}\right)}\left[1+\left(1-\frac{v}{c}\right)^2 - 2\left(1-\frac{v}{c}\right)\right]$$

$$\Delta\lambda=\frac{\lambda_F}{2\left(1-\frac{v}{c}\right)}\left[1+1-2\frac{v}{c}+\frac{v^2}{c^2}-2+2\frac{v}{c}\right]$$

$$\Delta\lambda=\frac{1}{\left(1-\frac{v}{c}\right)}\frac{\lambda_F}{2}\frac{v^2}{c^2} \qquad 10.4$$

Reference
http://www.wbabin.net/physics/faraj7.htm

§10 Ives-Stilwell (continuation)

The Doppler's effect transversal to the Undulating Relativity was obtained in the §2 as follows:

If the observer O', that sees the observer O, moves with the speed –v' in a negative way to the axis x', emits waves with the frequency y' and the speed c then the observer O according to 2.22 and $u'x'=-v'$ will measure waves of frequency y and speed c in a perpendicular plane to the movement of O' given by

$$y=y'\sqrt{1-\frac{v'^2}{c^2}} \qquad 2.25$$

For $u'x'=-v'$ we will have $ux=zero$ and $\sqrt{1-\frac{v'^2}{c^2}}\sqrt{1+\frac{v^2}{c^2}}=1$ with this we can write the relation between the transversal frequency $y=y_t$ and the source frequency $y'=y'_F$ like this

$$y_t=\frac{y'_F}{\sqrt{1+\frac{v^2}{c^2}}} \qquad 10.5$$

With $c=y_t\lambda_t=y'_F\lambda'_F$ we have the relation between the length of the transversal wave λ_t and the length of the source wave λ'_F

$$\lambda_t=\lambda'_F\sqrt{1+\frac{v^2}{c^2}} \qquad 10.6$$

The variation of the length of the transversal wave in the relation to the length of the source wave is:

$$\Delta\lambda_t=\lambda_t-\lambda'_F=\lambda'_F\sqrt{1+\frac{v^2}{c^2}}-\lambda'_F=\lambda'_F\left(\sqrt{1+\frac{v^2}{c^2}}-1\right)\cong\lambda'_F\left(1+\frac{v^2}{2c^2}-1\right)\cong\frac{\lambda'_F}{2}\frac{v^2}{c^2} \qquad 10.7$$

that is the same value gotten in the Theory of Special Relativity.

Applying 10.7 in 10.4 we have

$$\Delta\lambda=\frac{\Delta\lambda_t}{\left(1-\frac{v}{c}\right)} \qquad 10.8$$

With the equations 10.2 and 10.3 we can get the relations 10.9, 10.10, and 10.11 described as follows

$$\lambda_A=\lambda'_D\left(1-\frac{v}{c}\right)^2 \qquad 10.9$$

And from this we have the formula of speed $\frac{v}{c}=1-\sqrt{\frac{\lambda_A}{\lambda'_D}}$ 10.10

$$\lambda_F=\lambda'_F=\sqrt{\lambda_A\lambda'_D} \qquad 10.11$$

Applying 10.10 and 10.11 in 10.6 we have

$$\lambda_t=\sqrt{\lambda_A\lambda'_D}\sqrt{1+\left(1-\sqrt{\frac{\lambda_A}{\lambda'_D}}\right)^2} \qquad 10.12$$

From 10.8 and 10.12 we conclude that $\lambda_A \le \lambda_F \le \lambda_t \le \bar{\lambda} \le \lambda'_D$. 10.13

So that we the values of λ_A and λ'_D obtained from the Ives-Stiwell experience we can evaluate λ_t, λ_F, $\frac{v}{c}$ and conclude whether there is or not the space deformation predicted in the Theory of Special Relativity.

§11 Transformation of the power of a luminous ray between two referencials in the Special Theory of Relativity

The relationship within the power developed by the forces between two referencials is written in the Special Theory of the Relativity in the following way:

$$F'.u' = \frac{F.u - vFx}{\left(1 - \frac{vux}{c^2}\right)} \quad 11.1$$

The definition of the component of the force along the axis x is:

$$Fx = \frac{dpx}{dt} = \frac{d(mux)}{dt} = \frac{dm}{dt}ux + m\frac{dux}{dt} \quad 11.2$$

For a luminous ray, the principle of light speed constancy guarantees that the component ux of the light speed is also constant along its axis, thus

$\frac{x}{t} = \frac{dx}{dt} = ux =$ constant, demonstrating that in two $\frac{dux}{dt} = zero$ and $Fx = \frac{dm}{dt}ux$ 11.3

The formula of energy is $E = mc^2$ from where we have $\frac{dm}{dt} = \frac{1}{c^2}\frac{dE}{dt}$ 11.4

From the definition of energy we have $\frac{dE}{dt} = F.u$ that applying in 4 and 3 we have $Fx = F.u\frac{ux}{c^2}$ 11.5

Applying 5 in 1 we heve:

$$F'.u' = \frac{F.u - (F.u)\frac{vux}{c^2}}{\left(1 - \frac{vux}{c^2}\right)}$$

From where we find that $F'.u' = F.u$ or $\frac{dE'}{dt'} = \frac{dE}{dt}$ 11.6

A result equal to 5.3 of the Undulating Relativity that can be experimentally proven, considering the 'Sun' as the source.

§12 Linearity

The Theory of Undulating Relativity has as its fundamental axiom the necessity that inertial referentials be named exclusively as those ones in which a ray of light emitted in any direction from its origin spreads in a straight line, what is mathematically described by the formulae (1.13, 1.18, 8.6 e 8.7) of the Undulating Relativity:

$$\frac{x}{t} = \frac{dx}{dt} = ux, \frac{y}{t} = \frac{dy}{dt} = uy, \frac{z}{t} = \frac{dz}{dt} = uz \quad 1.13$$

$$\frac{x'}{t'} = \frac{dx'}{dt'} = u'x', \frac{y'}{t'} = \frac{dy'}{dt'} = u'y', \frac{z'}{t'} = \frac{dz'}{dt'} = u'z' \quad 1.18$$

Woldemar Voigt wrote in 1.887 the linear transformation between the referentials os the observers O e O' in the following way:

$$x = Ax' + Bt' \qquad 12.1$$

$$t = Ex' + Ft' \qquad 12.2$$

With the respective inverted equations:

$$x' = \frac{F}{AF - BE}x + \frac{-B}{AF - BE}t \qquad 12.3$$

$$t' = \frac{-E}{AF - BE}x + \frac{A}{AF - BE}t \qquad 12.4$$

Where A, B, E and F are constants and because of the symmetry we don't consider the terms with y, z and y', z'.

We know that x and x' are projections of the two rays of lights ct and ct' that spread with Constant speed c (due to the constancy principle of the Ray of light), emited in any direction from the origin of the respective inertials referential at the moment in which the origins are coincident and at the moment where:

t = t' = zero 12.5

because of this in the equation 12.2 at the moment where t' = zero we must have E = zero so that we also have t = zero, we can't assume that when t' = zero, x' also be equal to zero, because if the spreading happens in the plane y'z' we will have x' = zero plus $t' \neq zero$.

We should rewrite the corrected equations (E = zero):

$$x = Ax' + Bt' \qquad 12.6$$

$$t = Ft' \qquad 12.7$$

With the respective corrected inverted equations:

$$x' = \frac{x}{A} - \frac{Bt}{AF} \qquad 12.8$$

$$t' = \frac{t}{F} \qquad 12.9$$

If the spreading happens in the plane y' z' we have x' = zero and dividing 12.6 by 12.7 we have:

$$\frac{x}{t} = \frac{B}{F} = v \qquad 12.10$$

where v is the module of the speed in which the observer O sees the referential of the observer O' moving alongside the x axis in the positive way because the sign of the equation is positive.

If the spreading happens in the plane y z we have x = zero and dividing 12.8 by 12.9 we have:

$$\frac{x'}{t'} = -\frac{B}{A} = -v' \text{ or } \frac{B}{A} = v' \qquad 12.11$$

where v' is the module of the speed in which the observer O' sees the referential of the observer O moving alongside the x' axis in the negative way because the signal of the equation is negative.

The equation 1.6 describes the constancy principle of the speed of light that must be assumed by the equations 12.6 to 12.9:

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \qquad 1.6$$

Applying 12.6 and 12.7 in 1.6 we have:

$$(Ax' + Bt')^2 - c^2F^2t'^2 = x'^2 - c^2t'^2$$

From where we have:

$$\left(A^2x'^2\right)-c^2t'^2\left[F^2-\frac{B^2}{c^2}-\frac{2ABx'}{c^2t'}\right]=x'^2-c^2t'^2$$

where making A^2 = 1 in the brackets in arc and $\left[F^2-\frac{B^2}{c^2}-\frac{2ABx'}{c^2t'}\right]=1$ in the straight brackets we have the equality between both sides of the equal signal of the equation.

Appllying A = 1 in $\left[F^2-\frac{B^2}{c^2}-\frac{2ABx'}{c^2t'}\right]=1$ we have $F^2=1+\frac{B^2}{c^2}+\frac{2Bx'}{c^2t'}$ 12.12

Appllying A = 1 in 12.11 we have $\frac{B}{A}=\frac{B}{1}=B=v'$ 12.11

That applied in 12.12 suplies:

$$F=\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}}=F\left(x',t'\right) \quad 12.12$$

as F(x', t') is equal to the function F depending of the variables x' and t'.

Applying 12.8 and 12.9 in 1.6 we have:

$$x^2-c^2t^2=\left(\frac{x}{A}-\frac{Bt}{AF}\right)^2-c^2\frac{t}{F^2}^2$$

From where we have:

$$x^2-c^2t^2=\left(\frac{x^2}{A^2}\right)-c^2t^2\left[\frac{1}{F^2}-\frac{B^2}{A^2c^2F^2}+\frac{2Bx}{A^2c^2Ft}\right]$$

where making A^2 = 1 in the bracket in arc and $\left[\frac{1}{F^2}-\frac{B^2}{A^2c^2F^2}+\frac{2Bx}{A^2c^2Ft}\right]=1$ in the straight bracket we have the equality between both sides of the equal signal of the equation.

Applying A = 1 and 12.10 in $\left[\frac{1}{F^2}-\frac{B^2}{A^2c^2F^2}+\frac{2Bx}{A^2c^2Ft}\right]=1$ we have:

$$F=\frac{1}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}}}=F(x,t) \quad 12.13$$

as F(x, t) is equal to the function F depending on the variables x and t.

We must make the following naming according to 2.5 and 2.6:

$$K'=1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}\Rightarrow F=\sqrt{K'} \quad 12.14$$

$$K=1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}\Rightarrow F=\frac{1}{\sqrt{K}} \quad 12.15$$

As the equation to F(x', t') from 12.12 and F(x, t) from 12.13 must be equal, we have:

$$F=\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}}=\frac{1}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}}} \qquad 12.16$$

Thus:

$$\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}}\cdot\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}}=1 \text{ or } \sqrt{K}\cdot\sqrt{K'}=1 \qquad 12.17$$

Exactly equal to 1.10.

Rewriting the equations 12.6, 12.7, 12.8 and 12.9 according to the function of v, v' and F we have:

$$x=x'+v't' \qquad 12.6$$

$$t=Ft' \qquad 12.7$$

With the respective inverted corrected equations:

$$x'=x-vt \qquad 12.8$$

$$t'=\frac{t}{F} \qquad 12.9$$

We have the equations 12.6, 12.7, 12.8 and 12.9 finals replacing F by the corresponding formulae:

$$x=x'+v't' \qquad 12.6$$

$$t=t'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}} \qquad 12.7$$

With the respective inverted final equations:

$$x'=x-vt \qquad 12.8$$

$$t'=t\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}} \qquad 12.9$$

That are exactly the equations of the table I

As $v=\frac{B}{F}$ and $v'=B$ then the relations between v and v' are $v=\frac{v'}{F}$ or $v'=v.F$ 12.18

We will transform F (12.12) function of the elements v', x', and t' for F (12.13) function of the elements v, x and t, replacing in 12.12 the equations 12.8, 12.9 and 12.18:

$$F=\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}}=\sqrt{1+\frac{(vF)^2}{c^2}+\frac{2vF(x-vt)}{c^2\frac{t}{F}}}$$

$$F=\sqrt{1+\frac{v^2F^2}{c^2}+\frac{2vxF^2}{c^2t}-\frac{2v^2F^2}{c^2}}=\sqrt{1+\frac{2vxF^2}{c^2t}-\frac{v^2F^2}{c^2}}$$

$$F^2=1+\frac{2vxF^2}{c^2t}-\frac{v^2F^2}{c^2}\ \Rightarrow F^2+\frac{v^2F^2}{c^2}-\frac{2vxF^2}{c^2t}=1\Rightarrow F=\frac{1}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}}}$$

That is exactly the equation 12.13.

We will transform F (12.13) function of the elements v, x, and t for F (12.12) function of the elements v', x' and t', replacing in 12.13 the equations 12.6, 12.7 and 12.18:

$$F=\frac{1}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vx}{c^2t}}}=\frac{1}{\sqrt{1+\frac{1}{c^2}\left(\frac{v'}{F}\right)^2-\frac{2v'(x'+v't')}{c^2FFt'}}}=\frac{1}{\sqrt{1+\frac{v'^2}{c^2F^2}-\frac{2v'x'}{c^2t'F^2}-\frac{2v'^2}{c^2F^2}}}$$

$$F=\frac{1}{\sqrt{1-\frac{v'^2}{c^2F^2}-\frac{2v'x'}{c^2t'F^2}}}\Rightarrow F^2\left(1-\frac{v'^2}{c^2F2}-\frac{2v'x'}{c^2t'F^2}\right)=1\Rightarrow F=\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'x'}{c^2t'}}$$

That is exactly the equation 12.12.

We have to calculate the total diferential of F(x', t') (12.12):

$$dF=\frac{\partial F}{\partial x'}dx'+\frac{\partial F}{\partial t'}dt$$

as:

$$\frac{\partial F}{\partial x'}=\frac{1}{\sqrt{K'}}\frac{v'}{c^2t'}\ \text{and}\ \frac{\partial F}{\partial t'}=-\frac{1}{\sqrt{K'}}\frac{v'}{c^2t'}\frac{x'}{t'} \qquad 12.19$$

we have:

$$dF=\frac{1}{\sqrt{K'}}\frac{v'}{c^2t'}dx'-\frac{1}{\sqrt{K'}}\frac{v'}{c^2t'}\frac{x'}{t'}dt$$

where applying 1.18 we find:

$$dF=\frac{1}{\sqrt{K'}}\frac{v'}{c^2t'}dx'-\frac{1}{\sqrt{K'}}\frac{v'}{c^2t'}\frac{dx'}{dt}dt=o \qquad 12.20$$

From where we conclude that F function of x' and t' is a constant.

We have to calculate the total diferential of F(x, t) (12.13):

$$dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial t}dt$$

as:

$$\frac{\partial F}{\partial x}=\frac{1}{K^{\frac{3}{2}}}\frac{v}{c^2t}\ \text{and}\ \frac{\partial F}{\partial t}=-\frac{1}{K^{\frac{3}{2}}}\frac{v}{c^2t}\frac{x}{t} \qquad 12.21$$

we have:

$$dF=\frac{1}{K^{\frac{3}{2}}}\frac{v}{c^2t}dx-\frac{1}{K^{\frac{3}{2}}}\frac{v}{c^2t}\frac{x}{t}dt \qquad 12.22$$

where applying 1.13 we find:

$$dF=\frac{1}{K^{\frac{3}{2}}}\frac{v}{c^2t}dx-\frac{1}{K^{\frac{3}{2}}}\frac{v}{c^2t}\frac{dx}{dt}dt=o$$

From where we conclude that F function of x and t is a constant.

The equations 1.13 and 1.18 represent to the observers O and O' the principle of constancy of the light speed valid from infinitely small to the infinitely big and mean that in the Undulating Relativity the space and time are measure simultaneously. They shouldn't be interpreted with a dependency between space and time.

The time has its own interpretation that can be understood if we analyze to a determined observer the emission of two rays of light from the instant t=zero. If we add the times we get, for each ray of light, we will get a result without any use for the physics.

If in the instant t = t' = zero, the observer O' emits two rays of light, one alongside the axis x and the other alongside the axis y, after the interval of time t', the rays hit for the observer O', simultaneously, the points A_x and A_y to the distance ct' from the origin, although for the observer O, the points won't be hit simultaneously. For both rays of lights be simultaneous to both observers, they must hit the points that have the same radius in relation to the axis x and that provide the same time for both observers ($t_1 = t_2$ and $t'_1 = t'_2$), which means that only one ray of light is necessary to check the time between the referentials.
According to § 1, both referentials of the observers O and O' are inertial, thus the light spreads in a straight line according to what is demanded by the fundamental axiom of the Undulating Relativity § 12, because of this, the difference in velocities v and v' is due to only a difference in time between the referentials.

$$v=\frac{x-x'}{t} \qquad 1.2 \qquad\qquad v'=\frac{x-x'}{t'} \qquad 1.4$$

We can also relate na inertial referential for which the light spread in a straight line according to what is demanded by the fundamental axiom of the Undulating Relativity, with an accelerated moving referential for which the light spread in a curve line, considering that in this case the difference v and v' isn't due to only the difference of time between the referentials.

According to § 1, if the observer O at the instant t = t' = zero, emits a ray of light from the origin of its referential, after an interval of time t_1, the ray of light hits the point A_1 with coordinates (x_1, y_1, z_1, t_1) to the distance ct_1 of the origin of the observer O, then we have:

$$t'_1=t_1\sqrt{1+\frac{v^2}{c^2}-\frac{2vx_1}{c^2t_1}}$$

After hitting the point A_1 the ray of light still spread in the same direction and in the same way, after an interval of time t_2, the ray of light hits the point A_2 with coordinates ($x_1 + x_2$, $y_1 + y_2$, $z_1 + z_2$, $t_1 + t_2$) to the distance ct_2 to the point A_1, then we have:

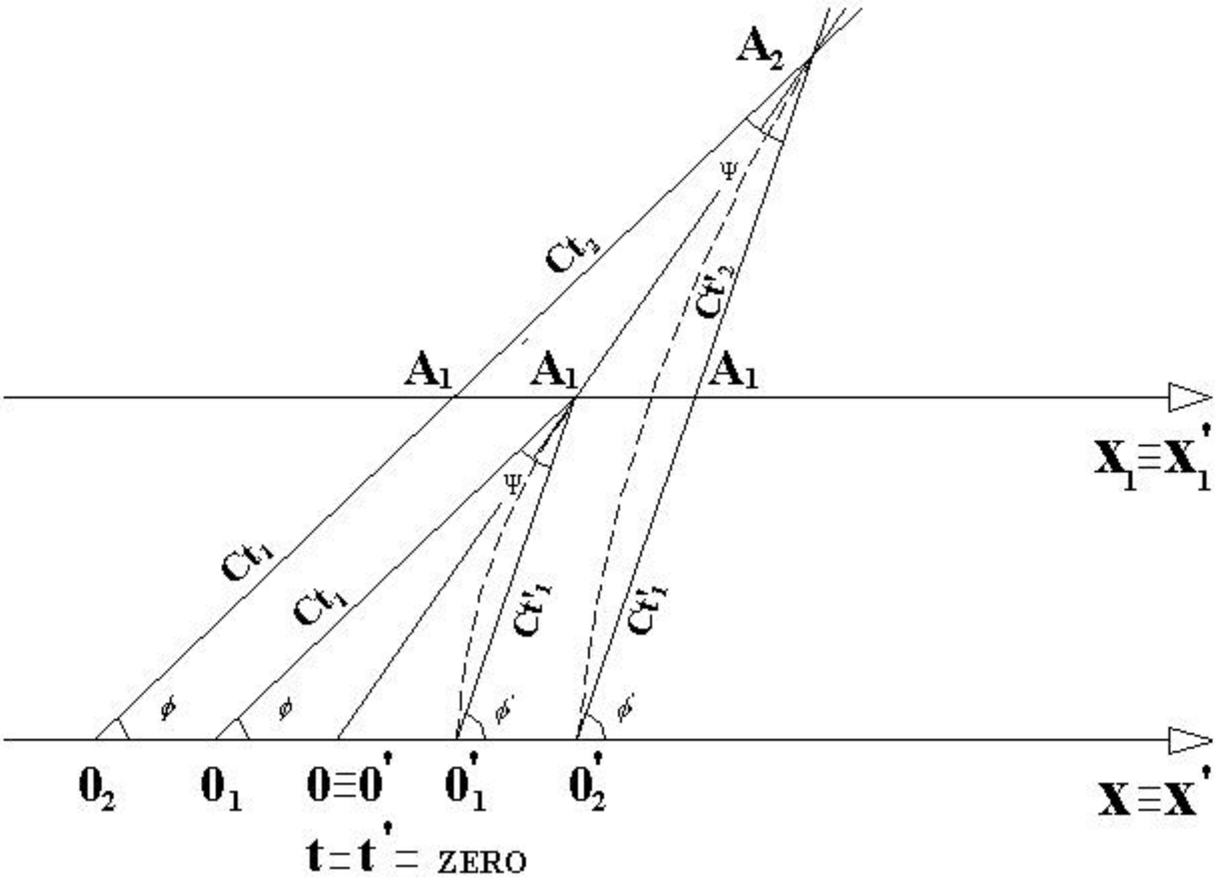
$$\frac{x}{t}=\frac{dx}{dt}=ux\Rightarrow\frac{x_1}{t_1}=\frac{x_2}{t_2}=ux\Rightarrow\sqrt{1+\frac{v^2}{c^2}-\frac{2vx_1}{c^2t_1}}=\sqrt{1+\frac{v^2}{c^2}-\frac{2vx_2}{c^2t_2}}=\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}$$

and with this we get:

$$t'_2=t_2\sqrt{1+\frac{v^2}{c^2}-\frac{2vx_2}{c^2t_2}}=t_2\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}$$

$$t'_1+t'_2=t_1\sqrt{1+\frac{v^2}{c^2}-\frac{2vx_1}{c^2t_1}}+t_2\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}=(t_1+t_2)\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}=(t_1+t_2)\sqrt{1+\frac{v^2}{c^2}-\frac{2v(x_1+x_2)}{c^2(t_1+t_2)}}$$

The geometry of space and time in the Undulating Relativity is summarized in the figure below that can be expanded to A_n points and several observers.



In the figure the angles have a relation $\psi=\phi'-\phi$ and are equal to the following segments:

O_1 to $O\equiv O'$ is equal to $O\equiv O'$ to O'_1 $\left(O_1\leftrightarrow O'_1=vt_1=v't'_1\right)$

O_2 to O_1 is equal to O'_1 to O'_2 $\left(O_2\leftrightarrow O'_2=v(t_1+t_2)=v'(t'_1+t'_2)\rightarrow vt_2=v't'_2=O_2\leftrightarrow O_1+O'_1\leftrightarrow O'_2\right)$

And are parallel to the following segments:

O_2 to A_2 is parallel to O_1 to A_1

O'_2 to A_2 is parallel to O'_1 to A_1

$X\equiv X'$ is parallel to $X_1\equiv X'_1$

The cosine of the angles of inclination ϕ and ϕ' to the rays for the observers O and O' according to 2.3 and 2.4 are:

$$u'x'=\frac{ux-v}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}}\Rightarrow\frac{u'x'}{c}=\frac{\frac{ux}{c}-\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}}\Rightarrow\cos\phi'=\frac{\cos\phi-v/c}{\sqrt{1+\frac{v^2}{c^2}-\frac{2v}{c}\cos\phi}}$$

$$\cos\phi'=\frac{\cos\phi-v/c}{\sqrt{K}} \qquad 12.23$$

And with this we have: $sen\phi'=\frac{sen\phi}{\sqrt{K}}$ 12.24

$$ux=\frac{u'x'+v'}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}}\Rightarrow\frac{ux}{c}=\frac{\frac{u'x'}{c}+\frac{v'}{c}}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}}\Rightarrow\cos\phi=\frac{\cos\phi'+v'/c}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'}{c}\cos\phi'}}$$

$$\cos\phi=\frac{\cos\phi'+v'/c}{\sqrt{K'}} \qquad 12.25$$

And with this we have $sen\phi = \frac{sen\phi'}{\sqrt{K'}}$ 12.26

The cosine of the angle ψ with intersection of rays equal to:

$$cos\psi = \frac{1 - \frac{vux}{c^2}}{\sqrt{K}} = \frac{1 + \frac{v'u'x'}{c^2}}{\sqrt{K'}} = \frac{1 - \frac{v}{c}cos\phi}{\sqrt{K}} = \frac{1 + \frac{v'}{c}cos\phi'}{\sqrt{K'}} \quad 12.27$$

And with this we have: $sen\psi = \frac{v}{c}\frac{sen\phi}{\sqrt{K}} = \frac{v'}{c}\frac{sen\phi'}{\sqrt{K'}}$ 12.28

The invariance of the $cos\psi$ shows the harmony of all adopted hypotheses for space and time in the Undulating Relativity.

The $cos\psi$ is equal to the Jacobians of the transformations for the space and time of the picture I, where the radicals

$\sqrt{K} = \sqrt{1 + \frac{v^2}{c^2} - \frac{2vx}{c^2 t}}$ and $\sqrt{K'} = \sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'x'}{c^2 t'}}$ are considered variables and are derived.

$$cos\psi = J = \frac{\partial x'^i}{\partial x^j} = \frac{\partial(x',y',z',t')}{\partial(x,y,z,t)} = \begin{vmatrix} 1 & 0 & 0 & -v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-v/c^2}{\sqrt{K}} & 0 & 0 & \frac{1}{\sqrt{K}}\left(1 + \frac{v^2}{c^2} - \frac{vx}{c^2 t}\right) \end{vmatrix} = \frac{1 - \frac{vx}{c^2 t}}{\sqrt{K}} = \frac{1 - \frac{vux}{c^2}}{\sqrt{K}} \quad 8.8$$

$$cos\psi = J' = \frac{\partial x^k}{\partial x'^l} = \frac{\partial(x,y,z,t)}{\partial(x',y',z',t')} = \begin{vmatrix} 1 & 0 & 0 & v' \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{v'/c^2}{\sqrt{K'}} & 0 & 0 & \frac{1}{\sqrt{K'}}\left(1 + \frac{v'^2}{c^2} + \frac{v'x'}{c^2 t'}\right) \end{vmatrix} = \frac{1 + \frac{v'x'}{c^2 t'}}{\sqrt{K'}} = \frac{1 + \frac{v'u'x'}{c^2}}{\sqrt{K'}} \quad 8.8$$

§13 Richard C. Tolman

The §4 Transformations of the Momenta of Undulating Relativity was developed based on the experience conducted by Lewis and Tolman, according to the reference [3]. Where the collision of two spheres preserving the principle of conservation of energy and the principle of conservation of momenta, shows that the mass is a function of the velocity according to:

$$m = \frac{m_0}{\sqrt{1 - \frac{(u)^2}{c^2}}}$$

where m_0 is the mass of the sphere when in resting position and $u = |\vec{u}| = \sqrt{\vec{u}\vec{u}}$ the module of its speed.

Analyzing the collision between two identical spheres when in relative resting position, that for the observer O' are named S'$_1$ and S'$_2$ are moving along the axis x' in the contrary way with the following velocities before the collision:

Table 1

Esphere S'$_1$	Esphere S'$_2$
$u'x'_1 = v'$	$u'x_2' = -v'$
$u'y'_1 = zero$	$u'y'_2 = zero$
$u'z'_1 = zero$	$u'z'_2 = zero$

For the observer O the same spheres are named S$_1$ and S$_2$ and have the velocities $(ux_1, ux_2, uy_i = uz_i = zero)$ before the collision calculated according to the table 2 as follows:

The velocity ux_1 of the sphere S_1 is equals to:

$$ux_1 = \frac{u'x'_1 + v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'_1}{c^2}}} = \frac{v' + v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'v'}{c^2}}} = \frac{2v'}{\sqrt{1 + \frac{3v'^2}{c^2}}} .$$

The transformation from v' to v according to 1.20 from Table 2 is:

$$v = \frac{v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'_1}{c^2}}} = \frac{v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'v'}{c^2}}} = \frac{v'}{\sqrt{1 + \frac{3v'^2}{c^2}}} .$$

That applied in ux_1 supplies:

$$ux_1 = 2\left(\frac{v'}{\sqrt{1 + \frac{3v'^2}{c^2}}}\right) = 2v$$

The velocity ux_2 of the sphere S_2 is equal to:

$$ux_2 = \frac{u'x'_2 + v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'_2}{c^2}}} = \frac{-v' + v'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'(-v')}{c^2}}} = zero$$

Table 2

Sphere S_1	Sphere S_2
$ux_1 = \frac{2v'}{\sqrt{1 + \frac{3v'}{c^2}}} = 2v$	$ux_2 = zero$
$uy_1 = zero$	$uy_2 = zero$
$uz_1 = zero$	$uz_2 = zero$

For the observers O and O' the two spheres have the same mass when in relative resting position. And for the observer O' the two spheres collide with velocities of equal module and opposite direction because of this the momenta $(p'_1 = p'_2)$ null themselves during the collision, forming for a brief time $(\Delta t')$ only one body of mass

$$m_0 = m'_1 + m'_2.$$

According to the principle of conservation of momenta for the observer O we will have to impose that the momenta before the collision are equal to the momenta after the collision, thus:

$$m_1 ux_1 + m_2 ux_2 = (m_1 + m_2) w$$

Where for the observer O, w is the arbitrary velocity that supposedly for a brief time (Δt) will also see the masses united $(m = m_1 + m_2)$ moving. As the masses m_i have different velocities and the masses vary according to their own velocities, this equation cannot be simplified algebraically, having this variation of masses:

To the left side of the equal sign in the equation we have:

$$u = ux_1 = 2v$$

$$m_1=\frac{m_0}{\sqrt{1-\frac{(u)^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{(ux_1)^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{(2v)^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{4v^2}{c^2}}}$$

$$u=ux_2=zero$$

$$m_2=\frac{m_0}{\sqrt{1-\frac{(u)^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{(ux_2)^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{(zero)^2}{c^2}}}=m_0$$

To the right side of the equal sign in the equation we have:

$$u=w$$

$$m_1=\frac{m_0}{\sqrt{1-\frac{(u)^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{(w)^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{w^2}{c^2}}}$$

$$m_2=\frac{m_0}{\sqrt{1-\frac{(u)^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{(w)^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{w^2}{c^2}}}$$

Applying in the equation of conservation of momenta we have:

$$m_1ux_1+m_2ux_2=(m_1+m_2)w=m_1w+m_2w$$

$$\frac{m_0}{\sqrt{1-\frac{4v^2}{c^2}}}2v+m_0.0=\frac{m_0}{\sqrt{1-\frac{w^2}{c^2}}}w+\frac{m_0}{\sqrt{1-\frac{w^2}{c^2}}}w$$

From where we have:

$$\frac{2m_0v}{\sqrt{1-\frac{4v^2}{c^2}}}=\frac{2m_0w}{\sqrt{1-\frac{w^2}{c^2}}}\Rightarrow\frac{v}{\sqrt{1-\frac{4v^2}{c^2}}}=\frac{w}{\sqrt{1-\frac{w^2}{c^2}}}$$

$$w=\frac{v}{\sqrt{1-\frac{3v^2}{c^2}}}$$

As $w\neq v$ for the observer O the masses united $(m=m_1+m_2)$ wouldn't move momentarily alongside to the observer O' which is conceivable if we consider that the instants $\Delta t\neq\Delta t'$ are different where supposedly the masses would be in a resting position from the point of view of each observer and that the mass acting with velocity 2v is bigger than the mass in resting position.

If we operate with these variables in line we would have:

$$m_1ux_1+m_2ux_2=(m_1+m_2)w=m_1w+m_2w$$

$$\frac{m_0}{\sqrt{1-\frac{1}{c^2}\left(\frac{2V}{\sqrt{1+\frac{3V}{c^2}}}\right)^2}}\frac{2V}{\sqrt{1+\frac{3V}{c^2}}}+m_0.0=\frac{m_0}{\sqrt{1-\frac{w^2}{c^2}}}w+\frac{m_0}{\sqrt{1-\frac{w^2}{c^2}}}w=\frac{2m_0w}{\sqrt{1-\frac{w^2}{c^2}}}$$

$$\frac{2m_0v'}{\sqrt{\left(1+\frac{3v'}{c^2}\right)\left(1-\frac{1}{c^2}\left(\frac{4v'^2}{\left(1+\frac{3v'}{c^2}\right)}\right)\right)}}=\frac{2m_0w}{\sqrt{1-\frac{w^2}{c^2}}}$$

$$\frac{2m_0v'}{\sqrt{1+\frac{3v'}{c^2}-\frac{4v'^2}{c^2}}}=\frac{2m_0w}{\sqrt{1-\frac{w^2}{c^2}}}$$

$$\frac{2m_0v'}{\sqrt{1-\frac{v'^2}{c^2}}}=\frac{2m_0w}{\sqrt{1-\frac{w^2}{c^2}}}$$

From where we conclude that $w=v'$ which must be equal to the previous value of w, that is:

$$w=v'=\frac{v}{\sqrt{1-\frac{3v^2}{c^2}}}$$

A relation between v and v' that is obtained from Table 2 when $ux_1=2v$ that corresponds for the observer O to the velocity acting over the sphere in resting position.

§14 Velocities composition

Reference – Millennium Relativity

URL: http://www.mrelativity.net/MBriefs/VComp_Sci_Estab_Way.htm

Let's write the transformations of Hendrik A. Lorentz for space and time in the Special Theory of Relativity:

$x'=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}$	14.1a	$x=\frac{x'+vt}{\sqrt{1-\frac{v^2}{c^2}}}$	14.3a
$y'=y$	14.1b	$y=y'$	14.3b
$z'=z$	14.1c	$z=z'$	14.3c
$t'=\frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$	14.2	$t=\frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$	14.4

From them we obtain the equations of velocity transformation:

$u'x'=\frac{ux-v}{1-\frac{vux}{c^2}}$	14.5a	$ux=\frac{u'x'+v}{1+\frac{vu'x'}{c^2}}$	14.6a
$u'y'=\frac{uy\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{vux}{c^2}}$	14.5b	$uy=\frac{u'y'\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{vu'x'}{c^2}}$	14.6b
$u'z'=\frac{uz\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{vux}{c^2}}$	14.5c	$uz=\frac{u'z'\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{vu'x'}{c^2}}$	14.6c

Let's consider that in relation to the observer O' an object moves with velocity:

$$u'x'=1{,}5.10^5\,km/s\,(=0{,}50c).$$

And that the velocity of the observer O’ in relation to the observer O is:

$$v = 1{,}5.10^5\,km/s\,(= 0{,}50c).$$

The velocity ux of the object in relation to the observer O must be calculated by the formula 14.6a:

$$ux = \frac{u'x' + v}{1 + \frac{v u'x'}{c^2}} = \frac{1{,}5.10^5 + 1{,}5.10^5}{1 + \frac{1{,}5.10^5 . 1{,}5.10^5}{\left(3{,}0.10^5\right)^2}} = 2{,}4.10^5\,km/s\,(= 0{,}80c).$$

Where we use $c = 3{,}0.10^5\,km/s\,(= 1{,}00c)$.

Considering that the object has moved during one second in relation to the observer O $(t = 1{,}00s)$ we can then with 14.2 calculate the time passed to the observer O’:

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t\left(1 - \frac{v ux}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1{,}00\left(1 - \frac{1{,}5.10^5 . 2{,}4.10^5}{\left(3{,}0.10^5\right)^2}\right)}{\sqrt{1 - \frac{\left(1{,}5.10^5\right)^2}{\left(3{,}0.10^5\right)^2}}} = \frac{0{,}60}{\sqrt{0{,}75}} \Rightarrow t' = 0{,}693s.$$

To the observer O the observer O’ is away the distance d given by the formula:

$$d = vt = 1{,}5.10^5 . 1{,}00 = 1{,}5.10^5\,km.$$

To the observer O’ the observer O is away the distance d’ given by the formula:

$$d' = vt' = 1{,}5.10^5 \frac{0{,}60}{\sqrt{0{,}75}} = 1{,}0392.10^5\,km.$$

To the distance of the object (d_O, d'_O) in relation to the observers O and O’ is given by the formulae:

$$d_O = ux\,t = 2{,}4.10^5 . 1{,}00 = 2{,}4.10^5\,km.$$

$$d'_O = u'x'\,t' = 1{,}5.10^5 . \frac{0{,}60}{\sqrt{0{,}75}} = 1{,}0392.10^5\,km.$$

To the observer O the distance between the object and the observer O’ is given by the formula:

$$\Delta d = d_O - d = 2{,}4.10^5 - 1{,}5.10^5 = 0{,}90.10^5\,km.$$

To the observer O the velocity of the object in relation to the observer O’ is given by:

$$\frac{\Delta d}{t} = \frac{0{,}90.10^5\,km}{1{,}00s} = 0{,}90.10^5\,km/s \text{ (=0,30c)}$$

Relating the times t and t’ using the formula $t' = t\sqrt{1 - \frac{v^2}{c^2}}$ is only possible and exclusively when $ux = v$ and $u'x' = zero$ what isn’t the case above, to make it possible to understand this we write the equations 14.2 and 14.4 in the formula below:

$t' = \frac{t\left(1 - \frac{v}{c}\cos\phi\right)}{\sqrt{1 - \frac{v^2}{c^2}}}$	14.2	$t = \frac{t'\left(1 + \frac{v}{c}\cos\phi'\right)}{\sqrt{1 - \frac{v^2}{c^2}}}$	14.4

Where $cos\phi=\frac{x}{ct}$ and $cos\phi'=\frac{x'}{ct'}$.

The equations above can be written as:

$$t'=f(t,\phi) \text{ e } t=f'(t',\phi') \qquad 14.7$$

In each referential of the observers O and O' the light propagation creates a sphere with radius ct and ct' that intercept each other forming a circumference that propagates with velocity c. The radius ct and ct' and the positive way of the axis x and x' form the angles ϕ and ϕ' constant between the referentials. If for the same pair of referentials te angles were variable the time would be alleatory and would become useless for the Physics. In the equation $t'=f(t,\phi)$ we have t' identical function of t and ϕ, if we have in it ϕ constant and t' varies according to t we get the common relation between the times t and t' between two referentials, however if we have t constant and t' varies according to ϕ we will have for each value of ϕ one value of t' and t between two different referentials, and this analysis is also valid for $t=f'(t',\phi')$.

Dividing 14.5a by c we have:

$$\frac{u'x'}{c}=\frac{\frac{ux}{c}-\frac{v}{c}}{1-\frac{vux}{c^2}}\Rightarrow cos\phi'=\frac{cos\phi-\frac{v}{c}}{1-\frac{v}{c}cos\phi}. \qquad 14.8$$

Where $cos\phi=\frac{x}{ct}=\frac{ux}{c}$ and $cos\phi'=\frac{x'}{ct'}=\frac{u'x'}{c}$.

Isolating the velocity we have:

$$\frac{v}{c}=\frac{(cos\phi-cos\phi')}{(1-cos\phi cos\phi')} \quad \text{or} \quad v=\frac{ux-u'x'}{1-\frac{uxu'x'}{c^2}} \qquad 14.9$$

From where we conclude that we must have angles ϕ and ϕ' constant so that we have the same velocity between the referentials.

This demand of constant angles between the referentials must solve the controversies of Herbert Dingle.

§15 Invariance

The transformations to the space and time of table I, group 1.2 plus 1.7, in the matrix form is written like this:

$$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & -v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \qquad 15.1$$

That written in the form below represents the same coordinate transformations:

$$\begin{bmatrix} x' \\ y' \\ z' \\ ct' \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & -v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ ct \end{bmatrix} \qquad 15.2$$

We call as:

$$x'=x^i=\begin{bmatrix} x' \\ y' \\ z' \\ ct' \end{bmatrix}=\begin{bmatrix} x'^1 \\ x'^2 \\ x'^3 \\ cx'^4 \end{bmatrix},\ \alpha=\alpha_{ij}=\begin{bmatrix} 1 & 0 & 0 & -v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K} \end{bmatrix},\ x=x^j=\begin{bmatrix} x \\ y \\ z \\ ct \end{bmatrix}=\begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ cx^4 \end{bmatrix} \qquad 15.3$$

That are the functions $x'^{i}=x'^{i}\left(x^{j}\right)=x'^{i}\left(x^{1},x^{2},x^{3},cx^{4}\right)=x'^{i}\left(x,y,z,ct\right)$ 15.4

That in the symbolic form is written:

$x'=\alpha.x$ or in the indexed form $x'^{i}=\sum_{j=1}^{4}\alpha_{ij}x^{j}\Rightarrow x'^{i}=\alpha_{ij}x^{j}$ 15.5

Where we use Einstein's sum convention.

The transformations to the space and time of table I, group 1.4 plus 1.8, in the matrix form is written:

$$\begin{bmatrix}x\\y\\z\\t\end{bmatrix}=\begin{bmatrix}1&0&0&v'\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K'}\end{bmatrix}\begin{bmatrix}x'\\y'\\z'\\t'\end{bmatrix} \quad 15.6$$

That written in the form below represents the same coordinate transformations:

$$\begin{bmatrix}x\\y\\z\\ct\end{bmatrix}=\begin{bmatrix}1&0&0&v'/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K'}\end{bmatrix}\begin{bmatrix}x'\\y'\\z'\\ct'\end{bmatrix} \quad 15.7$$

That we call as:

$$x=x^{k}=\begin{bmatrix}x\\y\\z\\ct\end{bmatrix}=\begin{bmatrix}x^{1}\\x^{2}\\x^{3}\\cx^{4}\end{bmatrix},\ \alpha'=\alpha'_{kl}=\begin{bmatrix}1&0&0&v'/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K'}\end{bmatrix},\ x'=x'^{l}=\begin{bmatrix}x'\\y'\\z'\\ct'\end{bmatrix}=\begin{bmatrix}x'^{1}\\x'^{2}\\x'^{3}\\cx'^{4}\end{bmatrix} \quad 15.8$$

That are the functions $x^{k}=x^{k}\left(x'^{l}\right)=x^{k}\left(x'^{1},x'^{2},x'^{3},cx'^{4}\right)=x^{k}\left(x',y',z',ct'\right)$ 15.9

That in the symbolic form is written:

$x=\alpha'.x'$ or in the indexed form $x^{k}=\sum_{l=1}^{4}\alpha'_{kl}x'^{l}\Rightarrow x^{k}=\alpha'_{kl}x'^{l}$ 15.10

Being $\sqrt{K}=\sqrt{1+\frac{v^{2}}{c^{2}}-\frac{2vx^{1}}{c^{2}x^{4}}}$ (1.7), $\sqrt{K'}=\sqrt{1+\frac{v'^{2}}{c^{2}}+\frac{2v'x'^{1}}{c^{2}x'^{4}}}$ (1.8) and $\sqrt{K}.\sqrt{K'}=1$ (1.10).

The transformation matrices $\alpha=\alpha_{ij}$ and $\alpha'=\alpha'_{kl}$ have the properties:

$$\alpha.\alpha'=\alpha_{ij}\alpha'_{kl}=\sum_{j=1}^{4}\alpha_{ij}\alpha'_{jl}=\begin{bmatrix}1&0&0&-v/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}1&0&0&v'/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K'}\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I=\delta^{j}_{l} \quad 15.11$$

$$\alpha^{t}\alpha'^{t}=\alpha_{ji}\alpha'_{lk}=\sum_{i=1}^{4}\alpha_{ji}\alpha'_{ik}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-v/c&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\v'/c&0&0&\sqrt{K'}\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I=\delta^{j}_{k} \quad 15.12$$

Where $\alpha^{t}=\alpha_{ji}$ is the transposed matrix of $\alpha=\alpha_{ij}$ and $\alpha'^{t}=\alpha'_{lk}$ is the transpose matrix of $\alpha'=\alpha'_{kl}$ and δ is the Kronecker's delta.

$$\alpha'\alpha=\alpha'_{kl}\alpha_{ij}=\sum_{l=1}^{4}\alpha'_{kl}\alpha_{lj}=\begin{bmatrix}1&0&0&v'/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K'}\end{bmatrix}\begin{bmatrix}1&0&0&-v/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K}\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I=\delta^{k}_{j} \quad 15.13$$

$$\alpha'^t\alpha^t=\alpha'_{lk}\alpha_{ji}=\sum_{k=1}^{4}\alpha'_{lk}\alpha_{ki}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\v'/c&0&0&\sqrt{K'}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-v/c&0&0&\sqrt{K}\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I=\delta^j_i \quad 15.14$$

Where $\alpha'^t=\alpha'_{lk}$ is the transposed matrix of $\alpha'=\alpha'_{kl}$ and $\alpha^t=\alpha_{ji}$ is the transposed matrix of $\alpha=\alpha_{ij}$ and δ is the Kronecker's delta.

Observation: the matrices α_{ij} and $\alpha'_{kl}$are inverse of one another but are not orthogonal, that is: $\alpha_{ji}\neq\alpha'_{kl}$ and $\alpha_{ij}\neq\alpha'_{lk}$.

The partial derivatives $\frac{\partial x'^i}{\partial x^j}$ of the total differential $dx'^i=\frac{\partial x'^i}{\partial x^j}dx^j$ of the coordinate components that correlate according to $x'^i=x'^i(x^j)$, where in the transformation matrix $\alpha=\alpha_{ij}$ the radical $\sqrt{K}$ is considered constant and equal to:

Table 10, partial derivatives of the coordinate components:

$\frac{\partial x'^i}{\partial x^j}=\frac{\partial x'^1}{\partial x^j}=$	$\frac{\partial x'^1}{\partial x^1}=1$	$\frac{\partial x'^1}{\partial x^2}=0$	$\frac{\partial x'^1}{\partial x^3}=0$	$\frac{\partial x'^1}{\partial x^4}=-\frac{v}{c}$
$\frac{\partial x'^i}{\partial x^j}=\frac{\partial x'^2}{\partial x^j}=$	$\frac{\partial x'^2}{\partial x^1}=0$	$\frac{\partial x'^2}{\partial x^2}=1$	$\frac{\partial x'^2}{\partial x^3}=0$	$\frac{\partial x'^2}{\partial x^4}=0$
$\frac{\partial x'^i}{\partial x^j}=\frac{\partial x'^3}{\partial x^j}=$	$\frac{\partial x'^3}{\partial x^1}=0$	$\frac{\partial x'^3}{\partial x^2}=0$	$\frac{\partial x'^3}{\partial x^3}=1$	$\frac{\partial x'^3}{\partial x^4}=0$
$\frac{\partial x'^i}{\partial x^j}=\frac{\partial x'^4}{\partial x^j}=$	$\frac{\partial x'^4}{\partial x^1}=0$	$\frac{\partial x'^4}{\partial x^2}=0$	$\frac{\partial x'^4}{\partial x^3}=0$	$\frac{\partial x'^4}{\partial x^4}=\sqrt{K}$

The total differential of the coordinates in the matrix form is equal to:

$$\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix}=\begin{bmatrix}1&0&0&-v/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}dx^1\\dx^2\\dx^3\\cdx^4\end{bmatrix} \quad 15.15$$

That we call as:

$$dx'=dx'^i=\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix},\ A=A^i_j=\frac{\partial x'^i}{\partial x^j}=\begin{bmatrix}1&0&0&-v/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K}\end{bmatrix},\ dx=dx^j=\begin{bmatrix}dx^1\\dx^2\\dx^3\\cdx^4\end{bmatrix} \quad 15.16$$

Then we have $dx'=Adx\Rightarrow dx'^i=\sum_{j=1}^{4}A^i_j dx^j\Rightarrow dx'^i=\frac{\partial x'^i}{\partial x^j}dx^j$ 15.17

The partial derivatives $\frac{\partial x^k}{\partial x'^l}$ of the total differential $dx^k=\frac{\partial x^k}{\partial x'^l}dx'^l$ of the coordinate components that correlate according to $x^k=x^k(x'^l)$, where in the transformation matrix $\alpha'=\alpha'_{kl}$ the radical $\sqrt{K'}$ is considered constant and equal to:

Table 11 partial derivatives of the coordinate components:

$\frac{\partial x^k}{\partial x'^l}=\frac{\partial x^1}{\partial x'^l}=$	$\frac{\partial x^1}{\partial x'^1}=1$	$\frac{\partial x^1}{\partial x'^2}=0$	$\frac{\partial x^1}{\partial x'^3}=0$	$\frac{\partial x^1}{\partial x'^4}=\frac{v'}{c}$
$\frac{\partial x^k}{\partial x'^l}=\frac{\partial x^2}{\partial x'^l}=$	$\frac{\partial x^2}{\partial x'^1}=0$	$\frac{\partial x^2}{\partial x'^2}=1$	$\frac{\partial x^2}{\partial x'^3}=0$	$\frac{\partial x^2}{\partial x'^4}=0$
$\frac{\partial x^k}{\partial x'^l}=\frac{\partial x^3}{\partial x'^l}=$	$\frac{\partial x^3}{\partial x'^1}=0$	$\frac{\partial x^3}{\partial x'^2}=0$	$\frac{\partial x^3}{\partial x'^3}=1$	$\frac{\partial x^3}{\partial x'^4}=0$
$\frac{\partial x^k}{\partial x'^l}=\frac{\partial x^4}{\partial x'^l}=$	$\frac{\partial x^4}{\partial x'^1}=0$	$\frac{\partial x^4}{\partial x'^2}=0$	$\frac{\partial x^4}{\partial x'^3}=0$	$\frac{\partial x^4}{\partial x'^4}=\sqrt{K'}$

The total differential of the coordinates in the matrix form is equal to:

$$\begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ cdx^4 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & v'/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K'} \end{bmatrix}\begin{bmatrix} dx'^1 \\ dx'^2 \\ dx'^3 \\ cdx'^4 \end{bmatrix} \qquad 15.18$$

That we call as:

$$dx=dx^k=\begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ cdx^4 \end{bmatrix},\ A'=A'^k_l=\frac{\partial x^k}{\partial x'^l}=\begin{bmatrix} 1 & 0 & 0 & v'/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K'} \end{bmatrix},\ dx'=dx'^l=\begin{bmatrix} dx'^1 \\ dx'^2 \\ dx'^3 \\ cdx'^4 \end{bmatrix} \qquad 15.19$$

Then we have: $dx=A'dx'\Rightarrow dx^k=\sum_{l=1}^{4}A'^k_l\,dx'^l\Rightarrow dx^k=\frac{\partial x^k}{\partial x'^l}dx'^l$ 15.20

The Jacobians of the transformations 15.15 and 15.18 are:

$$J=\frac{\partial x'^i}{\partial x^j}=\frac{\partial(x'^1,x'^2,x'^3,x'^4)}{\partial(x^1,x^2,x^3,x^4)}=\begin{vmatrix} 1 & 0 & 0 & -v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K} \end{vmatrix}=\sqrt{K} \qquad 15.21$$

$$J'=\frac{\partial x^k}{\partial x'^l}=\frac{\partial(x^1,x^2,x^3,x^4)}{\partial(x'^1,x'^2,x'^3,x'^4)}=\begin{vmatrix} 1 & 0 & 0 & v'/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K'} \end{vmatrix}=\sqrt{K'} \qquad 15.22$$

Where $\sqrt{K}=\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}$ (2.5), $\sqrt{K'}=\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'^1}{c^2}}$ (2.6) and $\sqrt{K}.\sqrt{K'}=1$ (1.23).

The matrices of the transformation A and A' also have the properties 15.11, 15.12, 15.13 and 15.14 of the matrices α and α'.

From the function $\phi=\phi(x^k)=\phi=\phi[x^k(x'^l)]$ where the coordinates correlate in the form $x^k=x^k(x'^l)$ we have $\frac{\partial\phi}{\partial x'^l}=\frac{\partial\phi}{\partial x^k}\frac{\partial x^k}{\partial x'^l}$ described as:

$$\frac{\partial\phi}{\partial x'^1}=\frac{\partial\phi}{\partial x^k}\frac{\partial x^k}{\partial x'^1}=\frac{\partial\phi}{\partial x^1}\frac{\partial x^1}{\partial x'^1}+\frac{\partial\phi}{\partial x^2}\frac{\partial x^2}{\partial x'^1}+\frac{\partial\phi}{\partial x^3}\frac{\partial x^3}{\partial x'^1}+\frac{\partial\phi}{\partial x^4}\frac{\partial x^4}{\partial x'^1}$$
$$\frac{\partial\phi}{\partial x'^2}=\frac{\partial\phi}{\partial x^k}\frac{\partial x^k}{\partial x'^2}=\frac{\partial\phi}{\partial x^1}\frac{\partial x^1}{\partial x'^2}+\frac{\partial\phi}{\partial x^2}\frac{\partial x^2}{\partial x'^2}+\frac{\partial\phi}{\partial x^3}\frac{\partial x^3}{\partial x'^2}+\frac{\partial\phi}{\partial x^4}\frac{\partial x^4}{\partial x'^2}$$
$$\frac{\partial\phi'}{\partial x'^3}=\frac{\partial\phi}{\partial x^k}\frac{\partial x^k}{\partial x'^3}=\frac{\partial\phi}{\partial x^1}\frac{\partial x^1}{\partial x'^3}+\frac{\partial\phi}{\partial x^2}\frac{\partial x^2}{\partial x'^3}+\frac{\partial\phi}{\partial x^3}\frac{\partial x^3}{\partial x'^3}+\frac{\partial\phi}{\partial x^4}\frac{\partial x^4}{\partial x'^3}$$
$$\frac{\partial\phi}{\partial x'^4}=\frac{\partial\phi}{\partial x^k}\frac{\partial x^k}{\partial x'^4}=\frac{\partial\phi}{\partial x^1}\frac{\partial x^1}{\partial x'^4}+\frac{\partial\phi}{\partial x^2}\frac{\partial x^2}{\partial x'^4}+\frac{\partial\phi}{\partial x^3}\frac{\partial x^3}{\partial x'^4}+\frac{\partial\phi'}{\partial x^4}\frac{\partial x^4}{\partial x'^4}$$

That in the matrix form and without presenting the function ϕ becomes:

$$\frac{\partial\phi}{\partial x'^l}=\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{\partial x'^4}\right]=\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{\partial x^4}\right]\begin{bmatrix}\frac{\partial x^1}{\partial x'^1}=1 & \frac{\partial x^1}{\partial x'^2}=0 & \frac{\partial x^1}{\partial x'^3}=0 & \frac{\partial x^1}{\partial x'^4}=v' \\ \frac{\partial x^2}{\partial x'^1}=0 & \frac{\partial x^2}{\partial x'^2}=1 & \frac{\partial x^2}{\partial x'^3}=0 & \frac{\partial x^2}{\partial x'^4}=0 \\ \frac{\partial x^3}{\partial x'^1}=0 & \frac{\partial x^3}{\partial x'^2}=0 & \frac{\partial x^3}{\partial x'^3}=1 & \frac{\partial x^3}{\partial x'^4}=0 \\ \frac{\partial x^4}{\partial x'^1}=\frac{v'}{c^2\sqrt{K'}} & \frac{\partial x^4}{\partial x'^2}=0 & \frac{\partial x^4}{\partial x'^3}=0 & \frac{\partial x^4}{\partial x'^4}=\frac{1}{\sqrt{K'}}\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'^1}{c^2}\right)\end{bmatrix}$$

Where replacing the items below:

$$\frac{\partial x^4}{\partial x'^1}=\frac{v'}{c^2\sqrt{K'}}=\frac{v}{c^2}$$

$$\frac{\partial x^1}{\partial x'^4}=v'=\frac{v}{\sqrt{K}}$$

$$\frac{\partial x^4}{\partial x'^4}=\frac{1}{\sqrt{K'}}\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'^1}{c^2}\right)=\frac{\partial x'^4}{\partial x^4}=\frac{1}{\sqrt{K}}\left(1+\frac{v^2}{c^2}-\frac{vux^1}{c^2}\right)$$

Observation: this last relation shows that the time varies in an equal form between the referentials.

We get:

$$\frac{\partial\phi}{\partial x'^l}=\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{\partial x'^4}\right]=\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{\partial x^4}\right]\begin{bmatrix}\frac{\partial x^1}{\partial x'^1}=1 & \frac{\partial x^1}{\partial x'^2}=0 & \frac{\partial x^1}{\partial x'^3}=0 & \frac{\partial x^1}{\partial x'^4}=\frac{v}{\sqrt{K}} \\ \frac{\partial x^2}{\partial x'^1}=0 & \frac{\partial x^2}{\partial x'^2}=1 & \frac{\partial x^2}{\partial x'^3}=0 & \frac{\partial x^2}{\partial x'^4}=0 \\ \frac{\partial x^3}{\partial x'^1}=0 & \frac{\partial x^3}{\partial x'^2}=0 & \frac{\partial x^3}{\partial x'^3}=1 & \frac{\partial x^3}{\partial x'^4}=0 \\ \frac{\partial x^4}{\partial x'^1}=\frac{v}{c^2} & \frac{\partial x^4}{\partial x'^2}=0 & \frac{\partial x^4}{\partial x'^3}=0 & \frac{\partial x^4}{\partial x'^4}=\frac{1}{\sqrt{K}}\left(1+\frac{v^2}{c^2}-\frac{vux^1}{c^2}\right)\end{bmatrix}$$

That is the group 8.1 plus 8.3 of the table 9, differential operators, in the matrix form.

From the function $\phi'=\phi'(x'^i)=\phi=\phi[x'^i(x^j)]$ where the coordinates correlate in the form $x'^i=x'^i(x^j)$ we have $\frac{\partial\phi'}{\partial x^j}=\frac{\partial\phi'}{\partial x'^i}\frac{\partial x'^i}{\partial x^j}$ described as:

$$\frac{\partial\phi}{\partial x^1}=\frac{\partial\phi}{\partial x'^i}\frac{\partial x'^i}{\partial x^1}=\frac{\partial\phi}{\partial x'^1}\frac{\partial x'^1}{\partial x^1}+\frac{\partial\phi}{\partial x'^2}\frac{\partial x'^2}{\partial x^1}+\frac{\partial\phi}{\partial x'^3}\frac{\partial x'^3}{\partial x^1}+\frac{\partial\phi}{\partial x'^4}\frac{\partial x'^4}{\partial x^1}$$
$$\frac{\partial\phi}{\partial x^2}=\frac{\partial\phi}{\partial x'^i}\frac{\partial x'^i}{\partial x^2}=\frac{\partial\phi}{\partial x'^1}\frac{\partial x'^1}{\partial x^2}+\frac{\partial\phi}{\partial x'^2}\frac{\partial x'^2}{\partial x^2}+\frac{\partial\phi}{\partial x'^3}\frac{\partial x'^3}{\partial x^2}+\frac{\partial\phi}{\partial x'^4}\frac{\partial x'^4}{\partial x^2}$$
$$\frac{\partial\phi}{\partial x^3}=\frac{\partial\phi}{\partial x'^i}\frac{\partial x'^i}{\partial x^3}=\frac{\partial\phi}{\partial x'^1}\frac{\partial x'^1}{\partial x^3}+\frac{\partial\phi}{\partial x'^2}\frac{\partial x'^2}{\partial x^3}+\frac{\partial\phi}{\partial x'^3}\frac{\partial x'^3}{\partial x^3}+\frac{\partial\phi}{\partial x'^4}\frac{\partial x'^4}{\partial x^3}$$
$$\frac{\partial\phi}{\partial x^4}=\frac{\partial\phi}{\partial x'^i}\frac{\partial x'^i}{\partial x^4}=\frac{\partial\phi}{\partial x'^1}\frac{\partial x'^1}{\partial x^4}+\frac{\partial\phi}{\partial x'^2}\frac{\partial x'^2}{\partial x^4}+\frac{\partial\phi}{\partial x'^3}\frac{\partial x'^3}{\partial x^4}+\frac{\partial\phi}{\partial x'^4}\frac{\partial x'^4}{\partial x^4}$$

That in the matrix form and without presenting the function ϕ becomes:

$$\frac{\partial\phi}{\partial x^j}=\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{\partial x^4}\right]=\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{\partial x'^4}\right]\begin{bmatrix}\frac{\partial x'^1}{\partial x^1}=1 & \frac{\partial x'^1}{\partial x^2}=0 & \frac{\partial x'^1}{\partial x^3}=0 & \frac{\partial x'^1}{\partial x^4}=-v\\ \frac{\partial x'^2}{\partial x^1}=0 & \frac{\partial x'^2}{\partial x^2}=1 & \frac{\partial x'^2}{\partial x^3}=0 & \frac{\partial x'^2}{\partial x^4}=0\\ \frac{\partial x'^3}{\partial x^1}=0 & \frac{\partial x'^3}{\partial x^2}=0 & \frac{\partial x'^3}{\partial x^3}=1 & \frac{\partial x'^3}{\partial x^4}=0\\ \frac{\partial x'^4}{\partial x^1}=\frac{-v}{c^2\sqrt{K}} & \frac{\partial x'^4}{\partial x^2}=0 & \frac{\partial x'^4}{\partial x^3}=0 & \frac{\partial x'^4}{\partial x^4}=\frac{1}{\sqrt{K}}\left(1+\frac{v^2}{c^2}-\frac{vux^1}{c^2}\right)\end{bmatrix}$$

Where replacing the items below:

$$\frac{\partial x'^1}{\partial x^4}=-v=\frac{-v'}{\sqrt{K'}}$$

$$\frac{\partial x'^4}{\partial x^1}=\frac{-v}{c^2\sqrt{K}}=\frac{-v'}{c^2}$$

$$\frac{\partial x'^4}{\partial x^4}=\frac{1}{\sqrt{K}}\left(1+\frac{v^2}{c^2}-\frac{vux^1}{c^2}\right)=\frac{\partial x^4}{\partial x'^4}=\frac{1}{\sqrt{K'}}\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'^1}{c^2}\right)$$

Observation: this last relation shows that the time varies in an equal form between the referentials.

We get:

$$\frac{\partial\phi}{\partial x^j}=\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{\partial x^4}\right]=\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{\partial x'^4}\right]\begin{bmatrix}\frac{\partial x'^1}{\partial x^1}=1 & \frac{\partial x'^1}{\partial x^2}=0 & \frac{\partial x'^1}{\partial x^3}=0 & \frac{\partial x'^1}{\partial x^4}=-\frac{v'}{\sqrt{K'}}\\ \frac{\partial x'^2}{\partial x^1}=0 & \frac{\partial x'^2}{\partial x^2}=1 & \frac{\partial x'^2}{\partial x^3}=0 & \frac{\partial x'^2}{\partial x^4}=0\\ \frac{\partial x'^3}{\partial x^1}=0 & \frac{\partial x'^3}{\partial x^2}=0 & \frac{\partial x'^3}{\partial x^3}=1 & \frac{\partial x'^3}{\partial x^4}=0\\ \frac{\partial x'^4}{\partial x^1}=\frac{-v'}{c^2} & \frac{\partial x'^4}{\partial x^2}=0 & \frac{\partial x'^4}{\partial x^3}=0 & \frac{\partial x'^4}{\partial x^4}=\frac{1}{\sqrt{K'}}\left(1+\frac{v'^2}{c^2}+\frac{v'u'x'^1}{c^2}\right)\end{bmatrix}$$

That is the group 8.2 plus 8.4 from the table 9, differential operators in the matrix form.

Applying 8.5 in 8.3 and in 8.4 we simplify these equations in the following way:

Table 9B, differential operators with the equations 8.3 and 8.4 simplified:

$\frac{\partial}{\partial x'^1}=\frac{\partial}{\partial x^1}+\frac{v}{c^2}\frac{\partial}{\partial x^4}$	8.1	$\frac{\partial}{\partial x^1}=\frac{\partial}{\partial x'^1}-\frac{v'}{c^2}\frac{\partial}{\partial x'^4}$	8.2
$\frac{\partial}{\partial x'^2}=\frac{\partial}{\partial x^2}$	8.1.1	$\frac{\partial}{\partial x^2}=\frac{\partial}{\partial x'^2}$	8.2.1
$\frac{\partial}{\partial x'^3}=\frac{\partial}{\partial x^3}$	8.1.2	$\frac{\partial}{\partial x^3}=\frac{\partial}{\partial x'^3}$	8.2.2
$\frac{-\partial}{c\partial x'^4}=\sqrt{K}\frac{-\partial}{c\partial x^4}$	8.3B	$\frac{-\partial}{c\partial x^4}=\sqrt{K'}\frac{-\partial}{c\partial x'^4}$	8.4B
$\frac{\partial}{\partial x^1}+\frac{ux^1}{c^2}\frac{\partial}{\partial x^4}=zero$	8.5	$\frac{\partial}{\partial x'^1}+\frac{u'x'^1}{c^2}\frac{\partial}{\partial x'^4}=zero$	8.5

The table 9B, in the matrix form becomes:

$$\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{-\partial}{c\partial x'^4}\right]=\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{-\partial}{c\partial x^4}\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-v/c&0&0&\sqrt{K}\end{bmatrix} \quad 15.23$$

$$\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{-\partial}{c\partial x^4}\right]=\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{-\partial}{c\partial x'^4}\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\v'/c&0&0&\sqrt{K'}\end{bmatrix} \quad 15.24$$

The squared matrices of the transformations above are transposed of the matrices A and A'.

Invariance of the Total Differential

In the observer O referential the total differential of a function $\phi(x^k)$ is equal to:

$$d\phi(x^k)=\frac{\partial\phi}{\partial x^k}dx^k=\frac{\partial\phi}{\partial x^1}dx^1+\frac{\partial\phi}{\partial x^2}dx^2+\frac{\partial\phi}{\partial x^3}dx^3+\frac{\partial\phi}{\partial x^4}dx^4=\left[\frac{\partial\phi}{\partial x^1}\frac{\partial\phi}{\partial x^2}\frac{\partial\phi}{\partial x^3}\frac{\partial\phi}{c\partial x^4}\right]\begin{bmatrix}dx^1\\dx^2\\dx^3\\cdx^4\end{bmatrix} \quad 15.25$$

Where the coordinates correlate with the ones from the observer O' according to $x^k=x^k(x'^l)$, replacing the transformations 15.24 and 15.18 and without presenting the function ϕ we have:

$$d\phi=\frac{\partial\phi}{\partial x^k}dx^k=\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{c\partial x'^4}\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-v'/c&0&0&\sqrt{K'}\end{bmatrix}\begin{bmatrix}1&0&0&v'/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K'}\end{bmatrix}\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix} \quad 15.26$$

The multiplication of the middle matrices supplies:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-v'/c&0&0&\sqrt{K'}\end{bmatrix}\begin{bmatrix}1&0&0&v'/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K'}\end{bmatrix}=\begin{bmatrix}1&0&0&v'/c\\0&1&0&0\\0&0&1&0\\-v'/c&0&0&1+\frac{2v'dx'^1}{c^2dx'^4}\end{bmatrix} \quad 15.27$$

Result that can be divided in two matrices:

$$\begin{bmatrix}1&0&0&v'/c\\0&1&0&0\\0&0&1&0\\-v'/c&0&0&1+\frac{2v'dx'^1}{c^2dx'^4}\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+\begin{bmatrix}0&0&0&v'/c\\0&0&0&0\\0&0&0&0\\-v'/c&0&0&\frac{2v'dx'^1}{c^2dx'^4}\end{bmatrix} \quad 15.28$$

That applied to the total differential supplies:

$$d\phi=\frac{\partial\phi}{\partial x^k}dx^k=\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{c\partial x'^4}\right]\left[\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+\begin{bmatrix}0&0&0&v'/c\\0&0&0&0\\0&0&0&0\\-v'/c&0&0&\frac{2v'dx'^1}{c^2dx'^4}\end{bmatrix}\right]\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix} \quad 15.29$$

Executing the operations of the second term we have:

$$\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{c\partial x'^4}\right]\begin{bmatrix}0&0&0&v'/c\\0&0&0&0\\0&0&0&0\\-v'/c&0&0&\frac{2v'dx'^1}{c^2dx'^4}\end{bmatrix}\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix}=-\frac{v'}{c^2}\frac{\partial}{\partial x'^4}dx'^1+v'\frac{\partial}{\partial x'^1}dx'^4+\frac{2v'}{c^2}\frac{dx'^1}{dx'^4}\frac{\partial}{\partial x'^4}dx'^4$$

Where applying 8.5 we have:

$$-\frac{v'}{c^2}\frac{\partial}{\partial x'^4}dx'^1+v'\left(-\frac{1}{c^2}\frac{dx'^1}{dx'^4}\frac{\partial}{\partial x'^4}\right)dx'^4+\frac{2v'}{c^2}\frac{dx'^1}{dx'^4}\frac{\partial}{\partial x'^4}dx'^4=zero$$

Then we have:

$$\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{c\partial x'^4}\right]\begin{bmatrix}0&0&0&v'/c\\0&0&0&0\\0&0&0&0\\-v'/c&0&0&\frac{2v'dx'^1}{c^2dx'^4}\end{bmatrix}\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix}=zero \quad 15.30$$

With this result we have in 15.29 the invariance of the total differential:

$$d\phi=\frac{\partial\phi}{\partial x^k}dx^k=\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{c\partial x'^4}\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix}=\frac{\partial\phi}{\partial x'^l}dx'^l=d\phi \quad 15.31$$

In the observer O' referential the total differential of a function $\phi(x'^i)$ is equal to:

$$d\phi(x'^i)=\frac{\partial\phi}{\partial x'^i}dx'^i=\frac{\partial\phi}{\partial x'^1}dx'^1+\frac{\partial\phi}{\partial x'^2}dx'^2+\frac{\partial\phi}{\partial x'^3}dx'^3+\frac{\partial\phi}{\partial x'^4}dx'^4=\left[\frac{\partial\phi}{\partial x'^1}\frac{\partial\phi}{\partial x'^2}\frac{\partial\phi}{\partial x'^3}\frac{\partial\phi}{c\partial x'^4}\right]\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix} \quad 15.32$$

Where the coordinates correlate with the ones from the observer O referential according to $x'^i=x'^i(x^j)$, replacing the transformations 15.23 and 15.15 and without presenting the function ϕ we have:

$$d\phi=\frac{\partial\phi}{\partial x'^i}dx'^i=\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{c\partial x^4}\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\v/c&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}1&0&0&-v/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}dx^1\\dx^2\\dx^3\\cdx^4\end{bmatrix} \quad 15.33$$

The multiplication of the middle matrices supplies:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\v/c&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}1&0&0&-v/c\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K}\end{bmatrix}=\begin{bmatrix}1&0&0&-v/c\\0&1&0&0\\0&0&1&0\\v/c&0&0&1-\frac{2vdx^1}{c^2dx^4}\end{bmatrix} \quad 15.34$$

Result that can be divided in two matrices:

$$\begin{bmatrix} 1 & 0 & 0 & -v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v/c & 0 & 0 & 1-\frac{2v\,dx^1}{c^2 dx^4} \end{bmatrix} = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -v/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ v/c & 0 & 0 & -\frac{2v\,dx^1}{c^2 dx^4} \end{bmatrix} \qquad 15.35$$

That applied to the total differential supplies:

$$d\phi = \frac{\partial\phi}{\partial x^i} dx^i = \begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{c\partial x^4} \end{bmatrix} \left[\begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -v/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ v/c & 0 & 0 & -\frac{2v\,dx^1}{c^2 dx^4} \end{bmatrix} \right] \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ c\,dx^4 \end{bmatrix} \qquad 15.36$$

Executing the operations of the second term we have:

$$\begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{c\partial x^4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -v/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ v/c & 0 & 0 & -\frac{2v\,dx^1}{c^2 dx^4} \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ c\,dx^4 \end{bmatrix} = \frac{v}{c^2}\frac{\partial}{\partial x^4}dx^1 - v\frac{\partial}{\partial x^1}dx^4 - \frac{2v}{c^2}\frac{dx^1}{dx^4}\frac{\partial}{\partial x^4}dx^4$$

Where applying 8.5 we have:

$$\frac{v}{c^2}\frac{\partial}{\partial x^4}dx^1 - v\left(-\frac{1}{c^2}\frac{dx^1}{dx^4}\frac{\partial}{\partial x^4}\right)dx^4 - \frac{2v}{c^2}\frac{dx^1}{dx^4}\frac{\partial}{\partial x^4}dx^4 = zero$$

Then we have:

$$\begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{c\partial x^4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -v/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ v/c & 0 & 0 & -\frac{2v\,dx^1}{c^2 dx^4} \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ c\,dx^4 \end{bmatrix} = zero \qquad 15.37$$

With this result we have in 15.36 the invariance of the total differential:

$$d\phi = \frac{\partial\phi}{\partial x^i} dx^i = \begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{c\partial x^4} \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ c\,dx^4 \end{bmatrix} = \frac{\partial\phi}{\partial x^j} dx^j = d\phi \qquad 15.38$$

Invariance of the Wave Equation

The wave equation to the observer O is equal to:

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial\phi^2}{\partial (x^4)^2} = \frac{\partial\phi^2}{\partial (x^1)^2} + \frac{\partial\phi^2}{\partial (x^2)^2} + \frac{\partial\phi^2}{\partial (x^3)^2} - \frac{1}{c^2}\frac{\partial\phi^2}{\partial (x^4)^2} = \begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{c\partial x^4} \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&-1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^3} \\ \frac{\partial}{c\partial x^4} \end{bmatrix} = 0 \qquad 15.39$$

Where applying 15.24 and the transposed from 15.24 we have:

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial\phi^2}{\partial (x^4)^2} = \begin{bmatrix} \frac{\partial}{\partial x'^1} & \frac{\partial}{\partial x'^2} & \frac{\partial}{\partial x'^3} & \frac{\partial}{c\partial x'^4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-v}{c} & 0 & 0 & \sqrt{K'} \end{bmatrix} \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&-1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \frac{-v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K'} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x'^1} \\ \frac{\partial}{\partial x'^2} \\ \frac{\partial}{\partial x'^3} \\ \frac{\partial}{c\partial x'^4} \end{bmatrix} \qquad 15.40$$

The multiplication of the three middle matrices supplies:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-v'}{c} & 0 & 0 & \sqrt{K'} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \frac{-v'}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{-v'}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-v'}{c} & 0 & 0 & -1-\frac{2v'u'x'^1}{c^2} \end{bmatrix} \quad 15.41$$

Result that can be divided in two matrices:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-v'}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-v'}{c} & 0 & 0 & -1-\frac{2v'u'x'^1}{c^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{-v'}{c} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-v'}{c} & 0 & 0 & \frac{-2v'u'x'^1}{c^2} \end{bmatrix} \quad 15.42$$

That applied in the wave equation supplies:

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial\phi^2}{\partial(x^4)^2} = \begin{bmatrix} \frac{\partial}{\partial x'^1} & \frac{\partial}{\partial x'^2} & \frac{\partial}{\partial x'^3} & \frac{\partial}{c\partial x'^4} \end{bmatrix} \left[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{-v'}{c} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-v'}{c} & 0 & 0 & \frac{-2v'u'x'^1}{c^2} \end{bmatrix} \right] \begin{bmatrix} \frac{\partial}{\partial x'^1} \\ \frac{\partial}{\partial x'^2} \\ \frac{\partial}{\partial x'^3} \\ \frac{\partial}{c\partial x'^4} \end{bmatrix} \quad 15.43$$

Executing the operations of the second term we have:

$$\begin{bmatrix} \frac{\partial}{\partial x'^1} & \frac{\partial}{\partial x'^2} & \frac{\partial}{\partial x'^3} & \frac{\partial}{c\partial x'^4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \frac{-v'}{c} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-v'}{c} & 0 & 0 & \frac{-2v'u'x'^1}{c^2} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x'^1} \\ \frac{\partial}{\partial x'^2} \\ \frac{\partial}{\partial x'^3} \\ \frac{\partial}{c\partial x'^4} \end{bmatrix} = -\frac{v'}{c^2}\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^4} - \frac{v'}{c^2}\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^4} - \frac{2v'u'x'^1}{c^2}\frac{1}{c^2}\frac{\partial^2}{\partial(x'^4)^2}$$

Executing the operations we have:

$$-\frac{2v'}{c^2}\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^4} - \frac{2v'u'x'^1}{c^2 c^2}\frac{\partial^2}{\partial(x'^4)^2}$$

Where applying 8.5 we have:

$$-\frac{2v'}{c^2}\left(-\frac{u'x'^1}{c^2}\frac{\partial}{\partial x'^4}\right)\frac{\partial}{\partial x'^4} - \frac{2v'u'x'^1}{c^2 c^2}\frac{\partial^2}{\partial(x'^4)^2} = zero$$

Then we have:

$$\begin{bmatrix} \frac{\partial}{\partial x'^1} & \frac{\partial}{\partial x'^2} & \frac{\partial}{\partial x'^3} & \frac{\partial}{c\partial x'^4} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \frac{-v'}{c} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-v'}{c} & 0 & 0 & \frac{-2v'u'x'^1}{c^2} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x'^1} \\ \frac{\partial}{\partial x'^2} \\ \frac{\partial}{\partial x'^3} \\ \frac{\partial}{c\partial x'^4} \end{bmatrix} = zero \quad 15.44$$

With this result we have in 15.43 the invariance of the wave equation:

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial\phi^2}{\partial\left(x^4\right)^2}=\begin{bmatrix}\frac{\partial}{\partial x'^1}&\frac{\partial}{\partial x'^2}&\frac{\partial}{\partial x'^3}&\frac{\partial}{c\partial x'^4}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x'^1}\\\frac{\partial}{\partial x'^2}\\\frac{\partial}{\partial x'^3}\\\frac{\partial}{c\partial x'^4}\end{bmatrix}=\nabla^2\phi'-\frac{1}{c^2}\frac{\partial\phi'^2}{\partial\left(x'^4\right)^2} \qquad 15.45$$

The wave equation to the observer O' is equal to:

$$\nabla^2\phi'-\frac{1}{c^2}\frac{\partial\phi'^2}{\partial\left(x'^4\right)^2}=\frac{\partial\phi'^2}{\partial\left(x'^1\right)^2}+\frac{\partial\phi'^2}{\partial\left(x'^2\right)^2}+\frac{\partial\phi'^2}{\partial\left(x'^3\right)^2}-\frac{1}{c^2}\frac{\partial\phi'^2}{\partial\left(x'^4\right)^2}=\begin{bmatrix}\frac{\partial}{\partial x'^1}&\frac{\partial}{\partial x'^2}&\frac{\partial}{\partial x'^3}&\frac{\partial}{c\partial x'^4}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x'^1}\\\frac{\partial}{\partial x'^2}\\\frac{\partial}{\partial x'^3}\\\frac{\partial}{c\partial x'^4}\end{bmatrix}=0 \qquad 15.46$$

Where applying 15.23 and the transposed from 15.23 we have:

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial\phi^2}{\partial\left(x^4\right)^2}=\begin{bmatrix}\frac{\partial}{\partial x^1}&\frac{\partial}{\partial x^2}&\frac{\partial}{\partial x^3}&\frac{\partial}{c\partial x^4}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\\frac{v}{c}&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0&\frac{v}{c}\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x^1}\\\frac{\partial}{\partial x^2}\\\frac{\partial}{\partial x^3}\\\frac{\partial}{c\partial x^4}\end{bmatrix} \qquad 15.47$$

The multiplication of the three middle matrices supplies:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\\frac{v}{c}&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0&\frac{v}{c}\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K}\end{bmatrix}=\begin{bmatrix}1&0&0&\frac{v}{c}\\0&1&0&0\\0&0&1&0\\\frac{v}{c}&0&0&-1+\frac{2vux^1}{c^2}\end{bmatrix} \qquad 15.48$$

Result that can be divided in two matrices:

$$\begin{bmatrix}1&0&0&\frac{v}{c}\\0&1&0&0\\0&0&1&0\\\frac{v}{c}&0&0&-1+\frac{2vux^1}{c^2}\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}+\begin{bmatrix}0&0&0&\frac{v}{c}\\0&0&0&0\\0&0&0&0\\\frac{v}{c}&0&0&\frac{2vux^1}{c^2}\end{bmatrix} \qquad 15.49$$

That applied in the wave equation supplies:

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial\phi^2}{\partial\left(x^4\right)^2}=\begin{bmatrix}\frac{\partial}{\partial x^1}&\frac{\partial}{\partial x^2}&\frac{\partial}{\partial x^3}&\frac{\partial}{c\partial x^4}\end{bmatrix}\left[\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}+\begin{bmatrix}0&0&0&\frac{v}{c}\\0&0&0&0\\0&0&0&0\\\frac{v}{c}&0&0&\frac{2vux^1}{c^2}\end{bmatrix}\right]\begin{bmatrix}\frac{\partial}{\partial x^1}\\\frac{\partial}{\partial x^2}\\\frac{\partial}{\partial x^3}\\\frac{\partial}{c\partial x^4}\end{bmatrix} \qquad 15.50$$

Executing the operations of the second term we have:

$$\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{c\partial x^4}\right]\begin{bmatrix}000 & \frac{v}{c}\\ 000 & 0\\ 000 & 0\\ \frac{v}{c}00 & \frac{2vux^1}{c^2}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x^1}\\ \frac{\partial}{\partial x^2}\\ \frac{\partial}{\partial x^3}\\ \frac{\partial}{\partial x^4}\end{bmatrix}=\frac{v}{c^2}\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^4}+\frac{v}{c^2}\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^4}+\frac{2vux^1}{c^2}\frac{\partial^2}{c^2\,\partial(x^4)^2}$$

Executing the operations we have:

$$\frac{2v}{c^2}\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^4}+\frac{2vux^1}{c^2}\frac{\partial^2}{c^2\,\partial(x^4)^2}$$

Where applying 8.5 we have:

$$\frac{2v}{c^2}\left(\frac{-ux^1}{c^2}\frac{\partial}{\partial x^4}\right)\frac{\partial}{\partial x^4}+\frac{2vux^1}{c^2}\frac{\partial^2}{c^2\,\partial(x^4)^2}=zero$$

Then we have:

$$\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{c\partial x^4}\right]\begin{bmatrix}000 & \frac{v}{c}\\ 000 & 0\\ 000 & 0\\ \frac{v}{c}00 & \frac{2vux^1}{c^2}\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x^1}\\ \frac{\partial}{\partial x^2}\\ \frac{\partial}{\partial x^3}\\ \frac{\partial}{\partial x^4}\end{bmatrix}=zero \qquad 15.51$$

Then in 15.50 we have the invariance of the wave equation:

$$\nabla^2\phi-\frac{1}{c^2}\frac{\partial\phi^2}{\partial(x^4)^2}=\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{c\partial x^4}\right]\begin{bmatrix}100 & 0\\ 010 & 0\\ 001 & 0\\ 000 & -1\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial x^1}\\ \frac{\partial}{\partial x^2}\\ \frac{\partial}{\partial x^3}\\ \frac{\partial}{c\partial x^4}\end{bmatrix}=\nabla^2\phi-\frac{1}{c^2}\frac{\partial\phi^2}{\partial(x^4)^2} \qquad 15.52$$

Invariance of the equations 8.5 of linear propagation

Replacing 2.4, 8.2, 8.4B in 8.5 we have:

$$\frac{\partial}{\partial x^1}+\frac{ux^1}{c^2}\frac{\partial}{\partial x^4}=\frac{\partial}{\partial x'^1}-\frac{v'}{c^2}\frac{\partial}{\partial x'^4}+\frac{1}{c^2}\frac{(u'x'^1+v')}{\sqrt{K'}}\sqrt{K'}\frac{\partial}{\partial x'^4}=zero$$

Executing the operations we have:

$$\frac{\partial}{\partial x^1}+\frac{ux^1}{c^2}\frac{\partial}{\partial x^4}=\frac{\partial}{\partial x'^1}-\frac{v'}{c^2}\frac{\partial}{\partial x'^4}+\frac{u'x'^1}{c^2}\frac{\partial}{\partial x'^4}+\frac{v'}{c^2}\frac{\partial}{\partial x'^4}=zero$$

That simplified supplies the invariance of the equation 8.5:

$$\frac{\partial}{\partial x^1}+\frac{ux^1}{c^2}\frac{\partial}{\partial x^4}=\frac{\partial}{\partial x'^1}+\frac{u'x'^1}{c^2}\frac{\partial}{\partial x'^4}=zero$$

Replacing 2.3, 8.1, 8.3B in 8.5 we have:

$$\frac{\partial}{\partial x'^1}+\frac{u'x'^1}{c^2}\frac{\partial}{\partial x'^4}=\frac{\partial}{\partial x^1}+\frac{v}{c^2}\frac{\partial}{\partial x^4}+\frac{1}{c^2}\frac{(ux^1-v)}{\sqrt{K}}\sqrt{K}\frac{\partial}{\partial x^4}=zero$$

Executing the operations we have:

$$\frac{\partial}{\partial x'^1}+\frac{u'x'^1}{c^2}\frac{\partial}{\partial x'^4}=\frac{\partial}{\partial x^1}+\frac{v}{c^2}\frac{\partial}{\partial x^4}+\frac{ux^1}{c^2}\frac{\partial}{\partial x^4}-\frac{v}{c^2}\frac{\partial}{\partial x^4}=zero$$

That simplified supplies the invariance of the equation 8.5:

$$\frac{\partial}{\partial x'^1}+\frac{u'x'^1}{c^2}\frac{\partial}{\partial x'^4}=\frac{\partial}{\partial x^1}+\frac{ux^1}{c^2}\frac{\partial}{\partial x^4}=zero$$

The table 4 in a matrix from becomes:

$$\begin{bmatrix} px'^1 \\ px'^2 \\ px'^3 \\ E'/c \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & -v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K} \end{bmatrix}\begin{bmatrix} px^1 \\ px^2 \\ px^3 \\ E/c \end{bmatrix} \quad 15.53$$

$$\begin{bmatrix} px^1 \\ px^2 \\ px^3 \\ E/c \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & v'/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K'} \end{bmatrix}\begin{bmatrix} px'^1 \\ px'^2 \\ px'^3 \\ E'/c \end{bmatrix} \quad 15.54$$

The table 6 in a matrix form becomes:

$$\begin{bmatrix} J'x'^1 \\ J'x'^2 \\ J'x'^3 \\ c\rho' \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & -v/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K} \end{bmatrix}\begin{bmatrix} Jx^1 \\ Jx^2 \\ Jx^3 \\ c\rho \end{bmatrix} \quad 15.55$$

$$\begin{bmatrix} Jx^1 \\ Jx^2 \\ Jx^3 \\ c\rho \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & v'/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K'} \end{bmatrix}\begin{bmatrix} J'x'^1 \\ J'x'^2 \\ J'x'^3 \\ c\rho' \end{bmatrix} \quad 15.56$$

Invariance of the Continuity Equation

The continuity equation to the observer O is equal to:

$$\nabla.J+\frac{\partial\rho}{\partial x^4}=\frac{\partial Jx^1}{\partial x^1}+\frac{\partial Jx^2}{\partial x^2}+\frac{\partial Jx^3}{\partial x^3}+\frac{\partial\rho}{\partial x^4}=\begin{bmatrix}\frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} & \frac{\partial}{c\partial x^4}\end{bmatrix}\begin{bmatrix} Jx^1 \\ Jx^2 \\ Jx^3 \\ c\rho \end{bmatrix}=zero \quad 15.57$$

Where replacing 15.24 and 15.56 we have:

$$\nabla.J+\frac{\partial\rho}{\partial x^4}=\begin{bmatrix}\frac{\partial}{\partial x'^1} & \frac{\partial}{\partial x'^2} & \frac{\partial}{\partial x'^3} & \frac{\partial}{c\partial x'^4}\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v'/c & 0 & 0 & \sqrt{K'} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & v'/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{K'} \end{bmatrix}\begin{bmatrix} J'x'^1 \\ J'x'^2 \\ J'x'^3 \\ c\rho' \end{bmatrix}=zero \quad 15.58$$

The product of the transformation matrices is given in 15.27 and 15.28 with this:

$$\nabla.J+\frac{\partial\rho}{\partial x^4}=\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{c\partial x'^4}\right]\left[\begin{bmatrix}1000\\0100\\0010\\0001\end{bmatrix}+\begin{bmatrix}0 & 00 & v'/c\\0 & 00 & 0\\0 & 00 & 0\\-v'/c & 00 & \frac{2v'u'x^1}{c^2}\end{bmatrix}\right]\begin{bmatrix}J'x^1\\J'x^2\\J'x^3\\c\rho'\end{bmatrix} \quad 15.59$$

Executing the operations of the second term we have:

$$\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{c\partial x'^4}\right]\begin{bmatrix}0 & 00 & v'/c\\0 & 00 & 0\\0 & 00 & 0\\-v'/c & 00 & \frac{2v'u'x^1}{c^2}\end{bmatrix}\begin{bmatrix}J'x^1\\J'x^2\\J'x^3\\c\rho'\end{bmatrix}=-\frac{v'}{c^2}\frac{\partial Jx'^1}{\partial x'^4}+\frac{v'\partial\rho'}{\partial x'^1}+\frac{2v'u'x^1}{c^2}\frac{\partial\rho'}{\partial x'^4}$$

Where replacing $Jx'^1=\rho' u'x'^1$ and 8.5 we have:

$$-\frac{v'u'x'^1}{c^2}\frac{\partial\rho'}{\partial x'^4}+v'\left(-\frac{u'x'^1}{c^2}\frac{\partial}{\partial x'^4}\right)\rho'+\frac{2v'u'x'^1}{c^2}\frac{\partial\rho'}{\partial x'^4}=zero$$

Then we have:

$$\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{c\partial x'^4}\right]\begin{bmatrix}0 & 00 & v'/c\\0 & 00 & 0\\0 & 00 & 0\\-v'/c & 00 & \frac{2v'u'x'^1}{c^2}\end{bmatrix}\begin{bmatrix}J'x^1\\J'x^2\\J'x^3\\c\rho'\end{bmatrix}=zero \quad 15.60$$

With this result we have in 15.59 the invariance of the continuity equation:

$$\nabla.J+\frac{\partial\rho}{\partial x^4}=\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{c\partial x'^4}\right]\begin{bmatrix}1000\\0100\\0010\\0001\end{bmatrix}\begin{bmatrix}J'x^1\\J'x^2\\J'x^3\\c\rho'\end{bmatrix}=\nabla.J'+\frac{\partial\rho'}{\partial x'^4} \quad 15.61$$

The continuity equation to the observer O' is equal to:

$$\nabla.J'+\frac{\partial\rho'}{\partial x'^4}=\frac{\partial J'x'^1}{\partial x'^1}+\frac{\partial J'x'^2}{\partial x'^2}+\frac{\partial J'x'^3}{\partial x'^3}+\frac{\partial\rho'}{\partial x'^4}=\left[\frac{\partial}{\partial x'^1}\frac{\partial}{\partial x'^2}\frac{\partial}{\partial x'^3}\frac{\partial}{c\partial x'^4}\right]\begin{bmatrix}J'x^1\\J'x^2\\J'x^3\\c\rho'\end{bmatrix}=zero \quad 15.62$$

Where replacing 15.23 and 15.55 we have:

$$\nabla.J'+\frac{\partial\rho'}{\partial x'^4}=\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{c\partial x^4}\right]\begin{bmatrix}1 & 00 & 0\\0 & 10 & 0\\0 & 01 & 0\\v/c & 00 & \sqrt{K}\end{bmatrix}\begin{bmatrix}100 & -v/c\\010 & 0\\001 & 0\\000 & \sqrt{K}\end{bmatrix}\begin{bmatrix}Jx^1\\Jx^2\\Jx^3\\c\rho\end{bmatrix}=zero \quad 15.63$$

The product of the transformation matrices is given in 15.34 and 15.35 then we have:

$$\nabla.J'+\frac{\partial\rho'}{\partial x'^4}=\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{c\partial x^4}\right]\left[\begin{bmatrix}1000\\0100\\0010\\0001\end{bmatrix}+\begin{bmatrix}0 & 00 & -v/c\\0 & 00 & 0\\0 & 00 & 0\\v/c & 00 & -\frac{2vux^1}{c^2}\end{bmatrix}\right]\begin{bmatrix}Jx^1\\Jx^2\\Jx^3\\c\rho\end{bmatrix} \quad 15.64$$

Executing the operations of the second term we have:

$$\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{c\partial x^4}\right]\begin{bmatrix}0&0&0&-v/c\\0&0&0&0\\0&0&0&0\\v/c&0&0&-\frac{2vux^1}{c^2}\end{bmatrix}\begin{bmatrix}Jx^1\\Jx^2\\Jx^3\\c\rho\end{bmatrix}=\frac{v}{c^2}\frac{\partial Jx^1}{\partial x^4}-\frac{v\partial\rho}{\partial x^1}-\frac{2vux^1}{c^2}\frac{\partial\rho}{\partial x^4}$$

Where replacing $Jx^1=\rho ux^1$ and 8.5 we have:

$$\frac{vux^1}{c^2}\frac{\partial\rho}{\partial x^4}-v\left(-\frac{ux^1}{c^2}\frac{\partial}{\partial x^4}\right)\rho-\frac{2vux^1}{c^2}\frac{\partial\rho}{\partial x^4}=zero$$

Then we have:

$$\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{c\partial x^4}\right]\begin{bmatrix}0&0&0&-v/c\\0&0&0&0\\0&0&0&0\\v/c&0&0&-\frac{2vux^1}{c^2}\end{bmatrix}\begin{bmatrix}Jx^1\\Jx^2\\Jx^3\\c\rho\end{bmatrix}=zero \qquad 15.65$$

With this result we have in 15.64 the invariance of the continuity equation:

$$\nabla.J'+\frac{\partial\rho'}{\partial x'^4}=\left[\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial x^3}\frac{\partial}{c\partial x^4}\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}Jx^1\\Jx^2\\Jx^3\\c\rho\end{bmatrix}=\nabla.J+\frac{\partial\rho}{\partial x^4} \qquad 15.66$$

Invariance of the line differential element:

That to the observer O is written this way:

$$(ds)^2=(dx^1)^2+(dx^2)^2+(dx^3)^2-(cdx^4)^2=\left[dx^1\ dx^2\ dx^3\ cdx^4\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}dx^1\\dx^2\\dx^3\\cdx^4\end{bmatrix} \qquad 15.67$$

Where replacing 15.18 and the transposed from 15.18 we have:

$$(ds)^2=\left[dx'^1\,dx'^2\,dx'^3\,cdx'^4\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\\frac{v'}{c}&0&0&\sqrt{K'}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0&\frac{v'}{c}\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K'}\end{bmatrix}\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix} \qquad 15.68$$

The multiplication of the three central matrices supplies:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\\frac{v'}{c}&0&0&\sqrt{K'}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0&\frac{v'}{c}\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K'}\end{bmatrix}=\begin{bmatrix}1&0&0&\frac{v'}{c}\\0&1&0&0\\0&0&1&0\\\frac{v'}{c}&0&0&-1-\frac{2v'dx'^1}{c^2dx'^4}\end{bmatrix} \qquad 15.69$$

Result that can be divided in two matrices:

$$\begin{bmatrix}1&0&0&\frac{v'}{c}\\0&1&0&0\\0&0&1&0\\\frac{v'}{c}&0&0&-1-\frac{2v'dx'^1}{c^2dx'^4}\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}+\begin{bmatrix}0&0&0&\frac{v'}{c}\\0&0&0&0\\0&0&0&0\\\frac{v'}{c}&0&0&-\frac{2v'dx'^1}{c^2dx'^4}\end{bmatrix} \qquad 15.70$$

That applied in the line differential element supplies:

$$(ds)^2 = \left[dx'^1\ dx'^2\ dx'^3\ cdx'^4\right]\left[\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}+\begin{bmatrix}0&0&0&\frac{v'}{c}\\0&0&0&0\\0&0&0&0\\\frac{v'}{c}&0&0&\frac{-2v'dx'^1}{c^2dx'^4}\end{bmatrix}\right]\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix} \quad 15.71$$

Executing the operations of the second term we have:

$$\left[dx'^1\ dx'^2\ dx'^3\ cdx'^4\right]\begin{bmatrix}0&0&0&\frac{v'}{c}\\0&0&0&0\\0&0&0&0\\\frac{v'}{c}&0&0&\frac{-2v'}{c^2}\frac{dx'^1}{dx'^4}\end{bmatrix}\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix}=\frac{v'dx'^1cdx'^4}{c}+cdx'^4\left(\frac{v'}{c}dx'^1-\frac{2v'}{c^2}\frac{dx'^1}{dx'^4}cdx'^4\right)=zero$$

Then we have:

$$\left[dx'^1\ dx'^2\ dx'^3\ cdx'^4\right]\begin{bmatrix}0&0&0&\frac{v'}{c}\\0&0&0&0\\0&0&0&0\\\frac{v'}{c}&0&0&\frac{-2v'}{c^2}\frac{dx'^1}{dx'^4}\end{bmatrix}\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix}=zero \quad 15.72$$

With this result we have in 15.71 the invariance of the line differential element:

$$(ds)^2 = \left[dx'^1\ dx'^2\ dx'^3\ cdx'^4\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix}=\left(dx'^1\right)^2+\left(dx'^2\right)^2+\left(dx'^3\right)^2-\left(cdx'^4\right)^2=(ds')^2 \quad 15.73$$

To the observer O' the line differential element is written this way:

$$(ds')^2=\left(dx'^1\right)^2+\left(dx'^2\right)^2+\left(dx'^3\right)^2-\left(cdx'^4\right)^2=\left[dx'^1\ dx'^2\ dx'^3\ cdx'^4\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}dx'^1\\dx'^2\\dx'^3\\cdx'^4\end{bmatrix} \quad 15.74$$

Where replacing 15.15 and the transposed from 15.15 we have:

$$(ds')^2=\left[dx^1dx^2dx^3cdx^4\right]\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\\frac{-v}{c}&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0&\frac{-v}{c}\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}dx^1\\dx^2\\dx^3\\cdx^4\end{bmatrix} \quad 15.75$$

The multiplication of the three central matrices supplies:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\\frac{-v}{c}&0&0&\sqrt{K}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0&\frac{-v}{c}\\0&1&0&0\\0&0&1&0\\0&0&0&\sqrt{K}\end{bmatrix}=\begin{bmatrix}1&0&0&\frac{-v}{c}\\0&1&0&0\\0&0&1&0\\\frac{-v}{c}&0&0&-1+\frac{2vdx^1}{c^2dx^4}\end{bmatrix} \quad 15.76$$

Result that can be divided in two matrices:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-v}{c} & 0 & 0 & -1+\frac{2v\,dx^1}{c^2dx^4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{-v}{c} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-v}{c} & 0 & 0 & \frac{2v\,dx^1}{c^2dx^4} \end{bmatrix} \quad 15.77$$

That applied in the line differential element supplies:

$$(ds)^2 = \begin{bmatrix} dx^1 & dx^2 & dx^3 & cdx^4 \end{bmatrix} \left[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{-v}{c} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-v}{c} & 0 & 0 & \frac{2v\,dx^1}{c^2dx^4} \end{bmatrix} \right] \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ cdx^4 \end{bmatrix} \quad 15.78$$

Executing the operations of the second term we have:

$$\begin{bmatrix} dx^1 & dx^2 & dx^3 & cdx^4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \frac{-v}{c} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-v}{c} & 0 & 0 & \frac{2v}{c^2}\frac{dx^1}{dx^4} \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ cdx^4 \end{bmatrix} = \frac{-v\,dx^1 cdx^4}{c} + cdx^4\left(\frac{-v}{c}dx^1 + \frac{2v}{c^2}\frac{dx^1}{dx^4}cdx^4 \right) = zero$$

Then we have:

$$\begin{bmatrix} dx^1 & dx^2 & dx^3 & cdx^4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \frac{-v}{c} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-v}{c} & 0 & 0 & \frac{2v}{c^2}\frac{dx^1}{dx^4} \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ cdx^4 \end{bmatrix} = zero \quad 15.79$$

With this result we have in 15.78 the invariance of the line differential element:

$$(ds)^2 = \begin{bmatrix} dx^1 & dx^2 & dx^3 & cdx^4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} dx^1 \\ dx^2 \\ dx^3 \\ cdx^4 \end{bmatrix} = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (cdx^4)^2 = (ds)^2 \quad 15.80$$

In §7 as a consequence of 5.3 we had the invariance of $E.u = E'.u'$ where now applying 7.3.1, 7.3.2, 7.4.1, 7.4.2 and the velocity transformation formulae from table 2 we have new relations between $E\lambda$ and $E'\lambda'$ distinct from 7.3 and 7.4 and with them we rewrite the table 7 in the form below:

Table 7B

$Ex' = \frac{Ex\sqrt{K}}{\left(1 - \frac{v}{ux}\right)}$	7.3B	$Ex = \frac{Ex'\sqrt{K'}}{\left(1 + \frac{v'}{ux'}\right)}$	7.4B
$E'y' = Ey\sqrt{K}$	7.3.1	$Ey = E'y'\sqrt{K'}$	7.4.1
$E'z' = Ez\sqrt{K}$	7.3.2	$Ez = E'z'\sqrt{K'}$	7.4.2
$B'x' = Bx$	7.5	$Bx = B'x'$	7.6
$B'y' = By + \frac{v}{c^2}Ez$	7.5.1	$By = B'y' - \frac{v'}{c^2}E'z'$	7.6.1
$B'z' = Bz - \frac{v}{c^2}Ey$	7.5.2	$Bz = B'z' + \frac{v'}{c^2}E'y'$	7.6.2
$By = -\frac{ux}{c^2}Ez$	7.9	$B'y' = -\frac{u'x'}{c^2}E'z'$	7.10
$Bz = \frac{ux}{c^2}Ey$	7.9.1	$B'z' = \frac{u'x'}{c^2}E'y'$	7.10.1
$\left(1 - \frac{v}{ux}\right)\left(1 + \frac{v'}{u'x'}\right) = 1$			

With the tables 7B and 9B we can have the invariance of all Maxwell's equations.

Invariance of the Gauss' Law for the electrical field:

$$\frac{\partial Ex'}{\partial x'} + \frac{\partial Ey'}{\partial y'} + \frac{\partial Ez'}{\partial z'} = \frac{\rho'}{\varepsilon_0} \quad 8.14$$

Where applying the tables 6, 7B and 9B we have:

$$\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\frac{Ex\sqrt{K}}{(1 - v/ux)} + \frac{\partial Ey\sqrt{K}}{\partial y} + \frac{Ez\sqrt{K}}{\partial z} = \frac{\rho\sqrt{K}}{\varepsilon_0}$$

Where simplifying and replacing 8.5 we have:

$$\left[\frac{\partial}{\partial x} + v\left(\frac{-1}{ux}\frac{\partial}{\partial x}\right)\right]\frac{Ex}{(1 - v/ux)} + \frac{\partial Ey}{\partial y} + \frac{Ez}{\partial z} = \frac{\rho}{\varepsilon_0}$$

That reordered supplies:

$$\left[\frac{\partial}{\partial x}\left(1 - \frac{v}{ux}\right)\right]\frac{Ex}{(1 - v/ux)} + \frac{\partial Ey}{\partial y} + \frac{Ez}{\partial z} = \frac{\rho}{\varepsilon_0}.$$

That simplified supplies the invariance of the Gauss' Law for the electrical field.

Invariance of the Gauss' Law for the magnetic field:

$$\frac{\partial Bx'}{\partial x'} + \frac{\partial By'}{\partial y'} + \frac{\partial Bz'}{\partial z'} = zero \quad 8.16$$

Where applying the tables 7B and 9B we have:

$$\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)Bx + \frac{\partial}{\partial y}\left(By + \frac{v}{c^2}Ez\right) + \frac{\partial}{\partial z}\left(Bz - \frac{v}{c^2}Ey\right) = 0$$

That reordered supplies:

$$\frac{\partial Bx}{\partial x}+\frac{\partial By}{\partial y}+\frac{\partial Bz}{\partial z}+\frac{v}{c^2}\left(\frac{\partial Ez}{\partial y}-\frac{\partial Ey}{\partial z}+\frac{\partial Bx}{\partial t}\right)=0$$

Where the term in parenthesis is the Faraday-Henry's Law (8.19) that is equal to zero from where we have the invariance of the Gauss' Law for the magnetic field.

Invariance of the Faraday-Henry's Law:

$$\frac{\partial Ey'}{\partial x'}-\frac{\partial Ex'}{\partial y'}=-\frac{\partial Bz'}{\partial t'} \qquad 8.18$$

Where applying the tables 7B and 9B we have:

$$\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)Ey\sqrt{K}-\frac{\partial}{\partial y}\frac{Ex\sqrt{K}}{(1-v/ux)}=-\sqrt{K}\frac{\partial}{\partial t}\left(Bz-\frac{v}{c^2}Ey\right)$$

That simplified and multiplied by $(1-v/ux)$ we have:

$$\frac{\partial Ey}{\partial x}\left(1-\frac{v}{ux}\right)-\frac{\partial Ex}{\partial y}=-\frac{\partial Bz}{\partial t}\left(1-\frac{v}{ux}\right)$$

Where executing the products and replacing 7.9.1 we have:

$$\frac{\partial Ey}{\partial x}-\frac{\partial Ex}{\partial y}=-\frac{\partial Bz}{\partial t}+\frac{v}{ux}\left(\frac{\partial Ey}{\partial x}+\frac{ux}{c^2}\frac{\partial Ey}{\partial t}\right)$$

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Faraday-Henry's Law.

Invariance of the Faraday-Henry's Law:

$$\frac{\partial Ez'}{\partial y'}-\frac{\partial Ey'}{\partial z'}=-\frac{\partial Bx'}{\partial t'} \qquad 8.20$$

Where applying the tables 7B and 9B we have:

$$\frac{\partial Ez}{\partial y}\sqrt{K}-\frac{\partial Ey}{\partial z}\sqrt{K}=-\sqrt{K}\frac{\partial Bx}{\partial t}$$

That simplified supplies the invariance of the Faraday-Henry's Law.

Invariance of the Faraday-Henry's Law:

$$\frac{\partial Ex'}{\partial z'}-\frac{\partial Ez'}{\partial x'}=-\frac{\partial By'}{\partial t'} \qquad 8.22$$

Where applying the tables 7B and 9B we have:

$$\frac{\partial}{\partial z}\frac{Ex\sqrt{K}}{(1-v/ux)}-\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)Ez\sqrt{K}=-\sqrt{K}\frac{\partial}{\partial t}\left(By+\frac{v}{c^2}Ez\right)$$

That simplified and multiplied by $(1-v/ux)$ we have:

$$\frac{\partial Ex}{\partial z}-\frac{\partial Ez}{\partial x}\left(1-\frac{v}{ux}\right)-\frac{v}{c^2}\frac{\partial Ez}{\partial t}\left(1-\frac{v}{ux}\right)=-\frac{\partial By}{\partial t}\left(1-\frac{v}{ux}\right)-\frac{v}{c^2}\frac{\partial Ez}{\partial t}\left(1-\frac{v}{ux}\right)$$

That simplifying and making the operations we have:

$$\frac{\partial Ex}{\partial z}-\frac{\partial Ez}{\partial x}=-\frac{\partial By}{\partial t}-\frac{v}{ux}\left(\frac{\partial Ez}{\partial x}-\frac{\partial By}{\partial t}\right)$$

Where applying 7.9 we have:

$$\frac{\partial Ex}{\partial z}-\frac{\partial Ez}{\partial x}=-\frac{\partial By}{\partial t}-\frac{v}{ux}\left(\frac{\partial Ez}{\partial x}+\frac{ux}{c^2}\frac{\partial Ez}{\partial t}\right).$$

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Faraday-Henry's Law.

Invariance of the Ampere-Maxwell's Law:

$$\frac{\partial By'}{\partial x'}-\frac{\partial Bx'}{\partial y'}=\mu_0 J'z'+\varepsilon_o\mu_0\frac{\partial Ez'}{\partial t'} \qquad 8.24$$

Where applying the tables 6, 7B and 9B we have:

$$\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)\left(By+\frac{v}{c^2}Ez\right)-\frac{\partial Bx}{\partial y}=\mu_0 Jz+\varepsilon_0\mu_0\sqrt{K}\frac{\partial}{\partial t}Ez/\sqrt{K}$$

That simplifying and making the operations we have:

$$\frac{\partial By}{\partial x}-\frac{\partial Bx}{\partial y}=\mu_0 Jz+\varepsilon_0\mu_0\frac{\partial Ez}{\partial t}+\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ez}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ez}{\partial t}-\frac{v}{c^2}\frac{\partial Ez}{\partial x}-\frac{v}{c^2}\frac{\partial By}{\partial t}-\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ez}{\partial t}$$

Where simplifying and applying 7.9 we have:

$$\frac{\partial By}{\partial x}-\frac{\partial Bx}{\partial y}=\mu_0 Jz+\varepsilon_0\mu_0\frac{\partial Ez}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ez}{\partial t}-\frac{v}{c^2}\frac{\partial Ez}{\partial x}-\frac{v}{c^2}\left(\frac{-ux}{c^2}\frac{\partial Ez}{\partial t}\right)$$

That reorganized supplies

$$\frac{\partial By}{\partial x}-\frac{\partial Bx}{\partial y}=\mu_0 Jz+\varepsilon_0\mu_0\frac{\partial Ez}{\partial t}-\frac{v}{c^2}\left(\frac{ux}{c^2}\frac{\partial Ez}{\partial t}+\frac{\partial Ez}{\partial x}\right)$$

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell's Law:

Invariance of the Ampere-Maxwell's Law:

$$\frac{\partial Bz'}{\partial y'}-\frac{\partial By'}{\partial z'}=\mu_0 J'x'+\varepsilon_o\mu_0\frac{\partial Ex'}{\partial t'} \qquad 8.26$$

Where applying the tables 6, 7B and 9B we have:

$$\frac{\partial}{\partial y}\left(Bz-\frac{v}{c^2}Ey\right)-\frac{\partial}{\partial z}\left(By+\frac{v}{c^2}Ez\right)=\mu_0(Jx-\rho v)+\varepsilon_0\mu_0\sqrt{K}\frac{\partial}{\partial t}\frac{Ex/\sqrt{K}}{(1-v/ux)}$$

Making the operations we have:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\mu_0 Jx+\frac{v}{c^2}\left(\frac{\partial Ey}{\partial y}+\frac{\partial Ez}{\partial z}-\mu_0 c^2\rho\right)+\varepsilon_0\mu_0\left(1+\frac{v^2}{c^2}-\frac{2vux}{c^2}\right)\frac{\partial Ex}{\partial t}\frac{1}{(1-v/ux)}$$

Replacing in the first parenthesis the Gauss' Law and multiplying by $\left(1-\frac{v}{ux}\right)$ we have:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\mu_0 Jx+\varepsilon_0\mu_0\frac{\partial Ex}{\partial t}+\frac{v}{ux}\left(\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}-\mu_0 Jx\right)-\frac{v}{c^2}\frac{\partial Ex}{\partial x}+\frac{v^2}{c^2}\left(\frac{1}{ux}\frac{\partial Ex}{\partial x}\right)+\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ex}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ex}{\partial t}$$

Where replacing $Jx=\rho ux$, 7.9.1, 7.9 and 8.5 we have:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\mu_0 Jx+\varepsilon_0\mu_0\frac{\partial Ex}{\partial t}+\frac{v}{ux}\left(\frac{ux}{c^2}\frac{\partial Ey}{\partial y}+\frac{ux}{c^2}\frac{\partial Ez}{\partial z}-\mu_0\rho ux\right)-\frac{v}{c^2}\frac{\partial Ex}{\partial x}+\frac{v^2}{c^2}\left(\frac{-1}{c^2}\frac{\partial Ex}{\partial t}\right)+\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ex}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ex}{\partial t}$$

That simplified supplies:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\mu_0 Jx+\varepsilon_0\mu_0\frac{\partial Ex}{\partial t}+\frac{v}{c^2}\left(\frac{\partial Ey}{\partial y}+\frac{\partial Ez}{\partial z}-\mu_0 c^2\rho\right)-\frac{v}{c^2}\frac{\partial Ex}{\partial x}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ex}{\partial t}$$

Replacing in the first parenthesis the Gauss' Law we have:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\mu_0 Jx+\varepsilon_0\mu_0\frac{\partial Ex}{\partial t}-\frac{v}{c^2}\frac{\partial Ex}{\partial x}-\frac{v}{c^2}\frac{\partial Ex}{\partial x}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ex}{\partial t}$$

That reorganized makes:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\mu_0 Jx+\varepsilon_0\mu_0\frac{\partial Ex}{\partial t}-\frac{2v}{c^2}\left(\frac{\partial Ex}{\partial x}+\frac{ux}{c^2}\frac{\partial Ex}{\partial t}\right)$$

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell's Law:

Invariance of the Ampere-Maxwell's Law:

$$\frac{\partial Bx'}{\partial z'}-\frac{\partial Bz'}{\partial x'}=\mu_o J'y'+\varepsilon_o\mu_o\frac{\partial Ey'}{\partial t'} \qquad 8.28$$

Where applying the tables 6, 7B and 9B we have:

$$\frac{\partial Bx}{\partial z}-\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)\left(Bz-\frac{v}{c^2}Ey\right)=\mu_0 Jy+\varepsilon_0\mu_0\sqrt{K}\frac{\partial}{\partial t}Ey\sqrt{K}$$

Making the operations we have:

$$\frac{\partial Bx}{\partial z}-\frac{\partial Bz}{\partial x}=\mu_0 Jy+\varepsilon_0\mu_0\frac{\partial Ey}{\partial t}+\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ey}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ey}{\partial t}-\frac{v}{c^2}\frac{\partial Ey}{\partial x}+\frac{v}{c^2}\frac{\partial Bz}{\partial t}-\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ey}{\partial t}$$

Where simplifying and applying 7.9.1 we have:

$$\frac{\partial Bx}{\partial z}-\frac{\partial Bz}{\partial x}=\mu_0 Jy+\varepsilon_0\mu_0\frac{\partial Ey}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ey}{\partial t}-\frac{v}{c^2}\frac{\partial Ey}{\partial x}+\frac{v}{c^2}\left(\frac{ux}{c^2}\frac{\partial Ey}{\partial t}\right)$$

That reorganized makes:

$$\frac{\partial Bx}{\partial z}-\frac{\partial Bz}{\partial x}=\mu_0 Jy+\varepsilon_0\mu_0\frac{\partial Ey}{\partial t}-\frac{v}{c^2}\left(\frac{ux}{c^2}\frac{\partial Ey}{\partial t}+\frac{\partial Ey}{\partial x}\right)$$

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell's Law:

Invariance of the Gauss' Law for the electrical field without electrical charge:

$$\frac{\partial Ex'}{\partial x'}+\frac{\partial Ey'}{\partial y'}+\frac{\partial Ez'}{\partial z'}=zero \qquad 8.30$$

Where applying the tables 7B and 9B we have:

$$\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)\frac{Ex\sqrt{K}}{(1-v/ux)}+\frac{\partial Ey\sqrt{K}}{\partial y}+\frac{\partial Ez\sqrt{K}}{\partial z}=zero$$

Where simplifying and replacing 8.5 we have:

$$\left[\frac{\partial}{\partial x}+v\left(\frac{-1}{ux}\frac{\partial}{\partial x}\right)\right]\frac{Ex}{(1-v/ux)}+\frac{\partial Ey}{\partial y}+\frac{\partial Ez}{\partial z}=zero$$

That reorganized makes:

$$\left[\frac{\partial}{\partial x}\left(1-\frac{v}{ux}\right)\right]\frac{Ex}{(1-v/ux)}+\frac{\partial Ey}{\partial y}+\frac{\partial Ez}{\partial z}=zero.$$

That simplified supplies the Gauss' Law for the electrical field without electrical charge.

Invariance of the Ampere-Maxwell's Law without electrical charge:

$$\frac{\partial By'}{\partial x'}-\frac{\partial Bx'}{\partial y'}=\varepsilon_o\mu_o\frac{\partial Ez'}{\partial t'} \qquad 8.40$$

Where applying the tables 7B and 9B we have:

$$\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)\left(By+\frac{v}{c^2}Ez\right)-\frac{\partial Bx}{\partial y}=\varepsilon_0\mu_0\sqrt{K}\frac{\partial}{\partial t}Ez\sqrt{K}$$

Making the operations we have:

$$\frac{\partial By}{\partial x}-\frac{\partial Bx}{\partial y}=\varepsilon_0\mu_0\frac{\partial Ez}{\partial t}+\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ez}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ez}{\partial t}-\frac{v}{c^2}\frac{\partial Ez}{\partial x}-\frac{v}{c^2}\frac{\partial By}{\partial t}-\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ez}{\partial t}$$

Where simplifying and applying 7.9 we have:

$$\frac{\partial By}{\partial x}-\frac{\partial Bx}{\partial y}=\varepsilon_0\mu_0\frac{\partial Ez}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ez}{\partial t}-\frac{v}{c^2}\frac{\partial Ez}{\partial x}-\frac{v}{c^2}\left(\frac{-ux}{c^2}\frac{\partial Ez}{\partial t}\right)$$

That reorganized makes:

$$\frac{\partial By}{\partial x}-\frac{\partial Bx}{\partial y}=\varepsilon_0\mu_0\frac{\partial Ez}{\partial t}-\frac{v}{c^2}\left(\frac{ux}{c^2}\frac{\partial Ez}{\partial t}+\frac{\partial Ez}{\partial x}\right)$$

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell's Law without electrical charge:

Invariance of the Ampere-Maxwell's Law without electrical charge:

$$\frac{\partial Bz'}{\partial y'}-\frac{\partial By'}{\partial z'}=\varepsilon_o\mu_o\frac{\partial Ex'}{\partial t'} \qquad 8.42$$

Where applying the tables 7B and 9B we have:

$$\frac{\partial}{\partial y}\left(Bz-\frac{v}{c^2}Ey\right)-\frac{\partial}{\partial z}\left(By+\frac{v}{c^2}Ez\right)=\varepsilon_0\mu_0\sqrt{K}\frac{\partial}{\partial t}\frac{Ex/K}{(1-v/ux)}$$

Making the operations we have:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\frac{v}{c^2}\left(\frac{\partial Ey}{\partial y}+\frac{\partial Ez}{\partial z}\right)+\varepsilon_0\mu_0\left(1+\frac{v^2}{c^2}-\frac{2vux}{c^2}\right)\frac{\partial Ex}{\partial t}\frac{1}{(1-v/ux)}$$

Replacing in the first parenthesis the Gauss' Law without electrical charge and multiplying by $(1-v/ux)$ we have:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\varepsilon_0\mu_0\frac{\partial Ex}{\partial t}+\frac{v}{ux}\left(\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}\right)-\frac{v}{c^2}\frac{\partial Ex}{\partial x}+\frac{v^2}{c^2}\left(\frac{1}{ux}\frac{\partial Ex}{\partial x}\right)+\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ex}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ex}{\partial t}$$

Where replacing 7.9, 7.9.1 and 8.5 we have:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\varepsilon_0\mu_0\frac{\partial Ex}{\partial t}+\frac{v}{ux}\left(\frac{ux}{c^2}\frac{\partial Ey}{\partial y}+\frac{ux}{c^2}\frac{\partial Ez}{\partial z}\right)-\frac{v}{c^2}\frac{\partial Ex}{\partial x}+\frac{v^2}{c^2}\left(\frac{-1}{c^2}\frac{\partial Ex}{\partial t}\right)+\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ex}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ex}{\partial t}$$

That simplified supplies:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\varepsilon_0\mu_0\frac{\partial Ex}{\partial t}+\frac{v}{c^2}\left(\frac{\partial Ey}{\partial y}+\frac{\partial Ez}{\partial z}\right)-\frac{v}{c^2}\frac{\partial Ex}{\partial x}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ex}{\partial t}$$

Replacing in the first parenthesis the Gauss' Law without electrical charge we have:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\varepsilon_0\mu_0\frac{\partial Ex}{\partial t}-\frac{v}{c^2}\frac{\partial Ex}{\partial x}-\frac{v}{c^2}\frac{\partial Ex}{\partial x}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ex}{\partial t}$$

That reorganized makes:

$$\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z}=\mu_0 Jx+\varepsilon_0\mu_0\frac{\partial Ex}{\partial t}-\frac{2v}{c^2}\left(\frac{\partial Ex}{\partial x}+\frac{ux}{c^2}\frac{\partial Ex}{\partial t}\right)$$

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell's Law without electrical charge:

Invariance of the Ampere-Maxwell's Law without electrical charge:

$$\frac{\partial Bx'}{\partial z'}-\frac{\partial Bz'}{\partial x'}=\varepsilon_o\mu_o\frac{\partial Ey'}{\partial t'} \qquad 8.44$$

Where applying the tables 6, 7B and 9B we have:

$$\frac{\partial Bx}{\partial z}-\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)\left(Bz-\frac{v}{c^2}Ey\right)=\varepsilon_0\mu_0\sqrt{K}\frac{\partial}{\partial t}Ey/\sqrt{K}$$

Making the operations we have:

$$\frac{\partial Bx}{\partial z}-\frac{\partial Bz}{\partial x}=\varepsilon_0\mu_0\frac{\partial Ey}{\partial t}+\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ey}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ey}{\partial t}-\frac{v}{c^2}\frac{\partial Ey}{\partial x}+\frac{v}{c^2}\frac{\partial Bz}{\partial t}-\frac{1}{c^2}\frac{v^2}{c^2}\frac{\partial Ey}{\partial t}$$

Where simplifying and applying 7.9.1 we have:

$$\frac{\partial Bx}{\partial z}-\frac{\partial Bz}{\partial x}=\varepsilon_0\mu_0\frac{\partial Ey}{\partial t}-\frac{1}{c^2}\frac{2vux}{c^2}\frac{\partial Ey}{\partial t}-\frac{v}{c^2}\frac{\partial Ey}{\partial x}+\frac{v}{c^2}\left(\frac{ux}{c^2}\frac{\partial Ey}{\partial t}\right)$$

That reorganized makes:

$$\frac{\partial Bx}{\partial z}-\frac{\partial Bz}{\partial x}=\varepsilon_0\mu_0\frac{\partial Ey}{\partial t}-\frac{v}{c^2}\left(\frac{ux}{c^2}\frac{\partial Ey}{\partial t}+\frac{\partial Ey}{\partial x}\right)$$

As the term in parenthesis is the equation 8.5 that is equal to zero then we have the invariance of the Ampere-Maxwell's Law without electrical charge:

§15 Invariance (continuation)

A function $f(\theta)=\mathrm{f}(\mathrm{kr}-wt)$ 2.19

Where the phase is equal to $\theta=(kr-wt)$ 15.81

In order to represent an undulating movement that goes on in one arbitrary direction must comply with the wave equation and because of this we have:

$$\frac{k}{r^2}\left[3r-\frac{(x^2+y^2+z^2)}{r}\right]\frac{\partial f(\theta)}{\partial\theta}+\frac{k^2}{r^2}(x^2+y^2+z^2)\frac{\partial^2 f(\theta)}{\partial\theta^2}-k^2\frac{\partial^2 f(\theta)}{\partial\theta^2}=zero \quad 15.82$$

That doesn't meet with the wave equation because the two last elements get nule but the first one doesn't. In order to overcome this problem we reformulate the phase θ of the function in the following way.

A unitary vector such as

$$\vec{n}=\cos\phi\,\vec{i}+\cos\alpha\,\vec{j}+\cos\beta\,\vec{k} \quad 15.83$$

where $\cos\phi=\frac{x}{r}=\frac{x}{ct}$, $\cos\alpha=\frac{y}{r}=\frac{y}{ct}$, $\cos\beta=\frac{z}{r}=\frac{z}{ct}$ 15.84

has the module equal to $n=|\vec{n}|=\sqrt{\vec{n}.\vec{n}}=\sqrt{\cos^2\phi+\cos^2\alpha+\cos^2\beta}=1.$ 15.85

Making the product

$$\vec{n}.\vec{R}=(\cos\phi\,\vec{i}+\cos\alpha\,\vec{j}+\cos\beta\,\vec{k}).(x\vec{i}+y\vec{j}+z\vec{k})=\cos\phi x+\cos\alpha y+\cos\beta z=\frac{x^2+y^2+z^2}{r}=\frac{r^2}{r}=r \quad 15.86$$

we have $r=\vec{n}.\vec{R}=\cos\phi x+\cos\alpha y+\cos\beta z$ that applied to the phase θ supplies a new phase

$$\Phi=(kr-wt)=(k\vec{n}.\vec{R}-wt)=(k\cos\phi x+k\cos\alpha y+k\cos\beta z-wt) \quad 15.87$$

with the same meaning of the previous phase $\theta=\Phi$.

Replacing $r=\vec{n}.\vec{R}=\cos\phi x+\cos\alpha y+\cos\beta z$ e $k=\frac{w}{c}$ in the phase θ multiplied by –1 we also get another phase in the form

$$\Phi=(-1)(kr-wt)=(wt-kr)=\left[w\left(t-\frac{r}{c}\right)\right]=\left[w\left(t-\frac{\cos\phi x+\cos\alpha y+\cos\beta z}{c}\right)\right] \quad 15.88$$

with the same meaning of the previous phase $(-1)\theta=\Phi$.

Thus we can write a new function as:

$$f(\Phi)=f\left[w\left(t-\frac{\cos\phi x+\cos\alpha y+\cos\beta z}{c}\right)\right] \quad 15.89$$

That replaced in the wave equation with the director cosine considered constant supplies:

$$\frac{\partial^2 f(\Phi)}{\partial\Phi^2}\frac{w^2}{c^2}\cos^2\phi+\frac{\partial^2 f(\Phi)}{\partial\Phi^2}\frac{w^2}{c^2}\cos^2\alpha+\frac{\partial^2 f(\Phi)}{\partial\Phi^2}\frac{w^2}{c^2}\cos^2\beta-\frac{\partial^2 f(\Phi)}{\partial\Phi^2}\frac{w^2}{c^2}=zero \quad 15.90$$

that simplified meets the wave equation.

The positive result of the phase Φ in the wave equation is an exclusive consequence of the director cosines being constant in the partial derivatives showing that the wave equation demands the propagation to have one steady direction in the space (plane wave).

For the observer O a source located in the origin of its referential produces in a random point located at the distance $r=ct=\sqrt{x^2+y^2+z^2}$ of the origin, an electrical field $\vec{E}$ described by:

$$\vec{E}=Ex\vec{i}+Ey\vec{j}+Ez\vec{k} \quad 15.91$$

Where the components are described as:

$$Ex=E_{xo}.f(\Phi)$$
$$Ey=E_{yo}.f(\Phi) \quad 15.92$$
$$Ez=E_{zo}.f(\Phi)$$

That applied in $\vec{E}$ supplies:

$$\vec{E}=E_{xo}f(\Phi)\vec{i}+E_{yo}f(\Phi)\vec{j}+E_{zo}f(\Phi)\vec{k}=\left[E_{xo}\vec{i}+E_{yo}\vec{j}+E_{zo}\right]f(\Phi)=\vec{E_o}f(\Phi). \quad 15.93$$

with module equal to $E=\sqrt{(E_{xo})^2+(E_{yo})^2+(E_{zo})^2}.f(\Phi)\Rightarrow E=E_o.f(\Phi)$ 15.94

Being $\vec{E_o}=E_{xo}\vec{i}+E_{yo}\vec{j}+E_{zo}\vec{k}$ 15.95

The maximum amplitude vector Constant with the components E_{xo}, E_{yo}, E_{zo} 15.96

And module $E_o=\sqrt{(E_{xo})^2+(E_{yo})^2+(E_{zo})^2}$ 15.97

Being $f(\Phi)$ a function with the phase Φ equal to15.87 or 15.88.

Deriving the component E_x in relation to x and t we have:

$$\frac{\partial Ex}{\partial x}=E_{xo}\frac{\partial f(\Phi)}{\partial\Phi}\frac{\partial\Phi}{\partial x}=E_{xo}\frac{\partial f(\Phi)}{\partial\Phi}\frac{\partial(kr-wt)}{\partial x}=E_{xo}\frac{\partial f(\Phi)}{\partial\Phi}\frac{kx}{r}=E_{xo}\frac{\partial f(\Phi)}{\partial\Phi}\frac{kx}{ct} \quad 15.98$$

$$\frac{\partial Ex}{\partial t}=E_{xo}\frac{\partial f(\Phi)}{\partial\Phi}\frac{\partial\Phi}{\partial t}=E_{xo}\frac{\partial f(\Phi)}{\partial\Phi}\frac{\partial(kr-wt)}{\partial t}=E_{xo}\frac{\partial f(\Phi)}{\partial\Phi}(-w) \quad 15.99$$

that applied in 8.5 supplies

$$\frac{\partial Ex}{\partial x}+\frac{x/t}{c^2}\frac{\partial Ex}{\partial t}=zero\Rightarrow E_{xo}\frac{\partial f(\Phi)}{\partial\Phi}\frac{\partial\Phi}{\partial x}+\frac{x/t}{c^2}E_{xo}\frac{\partial f(\Phi)}{\partial\Phi}\frac{\partial\Phi}{\partial t}=zero\Rightarrow E_{xo}\frac{\partial f(\Phi)}{\partial\Phi}\left(\frac{\partial\Phi}{\partial x}+\frac{x/t}{c^2}\frac{\partial\Phi}{\partial t}\right)=zero$$

$$E_{xo}\frac{\partial f(\Phi)}{\partial \Phi}\left(\frac{\partial \Phi}{\partial x}+\frac{x/t}{c^2}\frac{\partial \Phi}{\partial t}\right)=zero\Rightarrow\frac{\partial \Phi}{\partial x}+\frac{x/t}{c^2}\frac{\partial \Phi}{\partial t}=zero \qquad 15.100$$

demonstrating that it is the phase Φ that must comply with 8.5.

$$\frac{\partial \Phi}{\partial x}+\frac{x/t}{c^2}\frac{\partial \Phi}{\partial t}=zero\Rightarrow\frac{\partial (kr-wt)}{\partial x}+\frac{x/t}{c^2}\frac{\partial (kr-wt)}{\partial t}=zero\Rightarrow\frac{kx}{ct}+\frac{x/t}{c^2}(-w)=zero\Rightarrow\frac{x}{ct}\left(k-\frac{w}{c}\right)=zero$$

as $k=\frac{w}{c}$ then E_x complies with 8.5.

As the phase is the same for the components E_y and E_z then they also comply with 8.5.

As the phases for the observers O and O' are equal $(kr-wt)=(k'r'-w't')$ then the components of the observer O' also comply with 8.5.

$$\frac{\partial (kr-wt)}{\partial x}+\frac{x/t}{c^2}\frac{\partial (kr-wt)}{\partial t}=\frac{\partial (k'r'-w't')}{\partial x'}+\frac{x'/t'}{c^2}\frac{\partial (k'r'-w't')}{\partial t'}=zero \qquad 15.101$$

The components relatively to the observer O of the electrical field are transformed for the referential of the observer O' according to the tables 7, 7B and 8.

Applying in 8.5 a wave function written in the form:

$$\Psi=e^{i(kx-wt)}=e^{i\Phi}=\cos(kx-wt)+i\sin(kx-wt)=\cos\Phi+i\sin\Phi \qquad 15.102$$

where $i=\sqrt{-1}$.

Deriving we have:

$$\frac{\partial \Psi}{\partial x}=-ksen\Phi+kicos\Phi \text{ end } \frac{\partial \Psi}{\partial t}=wsen\Phi-wicos\Phi \qquad 15.103$$

or $\frac{\partial \Psi}{\partial x}=ke^{i\Phi}$ and $\frac{\partial \Psi}{\partial t}=-we^{i\Phi}$ 15.104

That applied in 8.5 supplies:

$$\frac{\partial \Psi}{\partial x}+\frac{x/t}{c^2}\frac{\partial \Psi}{\partial t}=zero\Rightarrow(-ksen\Phi+kicos\Phi)+\frac{x/t}{c^2}(wsen\Phi-wicos\Phi)=zero$$

that is equal to:

$$\left(-k+\frac{xw}{c^2t}\right)\sin\Phi+\left(ki-\frac{xwi}{c^2t}\right)\cos\Phi=zero$$

or $\frac{\partial \Psi}{\partial x}+\frac{x/t}{c^2}\frac{\partial \Psi}{\partial t}=zero\Rightarrow\left(ke^{i\Phi}\right)+\frac{x/t}{c^2}\left(-we^{i\Phi}\right)=zero$

where we must have the coefficients equal to zero so that we get na identity, then:

$$-k+\frac{xw}{c^2t}=zero\Rightarrow k=\frac{xw}{c^2t}$$

$$ki-\frac{xwi}{c^2t}=zero\Rightarrow k=\frac{xw}{c^2t}$$

$$\left(ke^{i\Phi}\right)+\frac{x/t}{c^2}\left(-we^{i\Phi}\right)=zero\Rightarrow k=\frac{xw}{c^2t}$$

Where applying $w=ck$ we have:

$$k=\frac{xw}{c^2t}=\frac{xck}{c^2t}\Rightarrow\frac{x}{t}=c$$

Then to meet with the equation 8.5 we must have a wave propagation along the axis x with the speed c.

If we apply $w=uk$ and $v=\frac{x}{t}$ we have:

$$k=\frac{xw}{c^2t}=\frac{vuk}{c^2}\Rightarrow u=\frac{c^2}{v}.$$

A result also gotten from the Louis de Broglie's wave equation.

§16 Time and Frequency

Considering the Doppler effect as a law of physics.

We can define a clock as any device that produces a frequency of identical events in a series possible to be enlisted and added in such a way that a random event *n* of a device will be identical to any event in the series of events produced by a replica of this device when the events are compared in a relative resting position.

The cyclical movement of a clock in a resting position according to the observer O referential sets the time in this referential and the cyclical movement of the arms of a clock in a resting position according to the observer O' sets the time in this referential. The formulas of time transformation 1.7 and 1.8 relate the times between the referentials in relative movement thus, relate movements in relative movement.

The relative movement between the inertial referentials produces the Doppler effect that proves that the frequency varies with velocity and as the frequency can be interpreted as being the frequency of the cyclical movement of the arms of a clock then the time varies in the same proportion that varies the frequency with the relative movement that is, it is enough to replace the time t and t' in the formulas 1.7 and 1.8 by the frequencies y and y' to get the formulas of frequency transformation, then:

$t'=t\sqrt{K}\Rightarrow y'=y\sqrt{K}$ 1.7 becomes 2.22

$t=t'\sqrt{K'}\Rightarrow y=y'\sqrt{K'}$ 1.8 becomes 2.22

The Galileo's transformation of velocities $\vec{u}'=\vec{u}-\vec{v}$ between two inertial referentials presents intrinsically three defects that can be described this way:

a) The Galileo's transformation of velocity to the axis x is $u'_x=u_x-v$. In that one if we have $u_x=c$ then $u'_x=c-v$ and if we have $u'_x=c$ then $u_x=c+v$. As both results are not simultaneously possible or else we have $u_x=c$ or $u'_x=c$ then the transformation doesn't allow that a ray of light be simultaneously observed by the observers O and O' what shows the privilege of an observer in relation to the other because each observer can only see the ray of light running in its own referential (intrinsic defect to the classic analysis of the Sagnac's effect).

b) It cannot also comply to Newton's first law of inertia because a ray of light emitted parallel to the axis x from the origin of the respective inertial referentials at the moment that the origins are coincident and at the moment in which t = t' = zero will have by the Galileo's transformation the velocity c of light altered by $\pm v$ to the referentials, on the contrary of the inertial law that wouldn't allow the existence of a variation in velocity because there is no external action acting on the ray of light and because of this both observers should see the ray of light with velocity c.

c) As it considers the time as a constant between the referentials it doesn't produce the temporal variation between the referentials in movement as it is required by the Doppler effect.

The principle of constancy of light velocity is nothing but a requirement of the Newton's first law, the inertia law.

Newton's first law, the inertia law, is introduced in Galileo's transformation when the principle of constancy of light velocity is applied in Galileo's transformation providing the equation of tables 1 and 2 of the Undulating Relativity that doesn't have the three defects described.
The time and velocity equations of tables 1 and 2 can be written as:

$$t' = t\sqrt{1+\frac{v^2}{c^2}-\frac{2v}{c}\cos\phi} \quad 1.7$$

$$v' = \frac{v}{\sqrt{1+\frac{v^2}{c^2}-\frac{2v}{c}\cos\phi}} \quad 1.15$$

$$t = t'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'}{c}\cos\phi'} \quad 1.8$$

$$v = \frac{v'}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'}{c}\cos\phi'}} \quad 1.20$$

The distance d between the referentials is equal to the product of velocity by time this way:

$$d = vt = v't' \quad 1.9$$

It doesn't depend on the propagation angle of the ray of light, being exclusively a function of velocity and time, that is, the propagation angle of the ray of light, only alters between the inertial referential the proportion between time and velocity, keeping the distance constant in each moment, to any propagation angle.

The equations above in a function form are written as:

$$d = e(v,t) = e'(v',t') \quad 1.9$$

$$t' = f(v,t,\phi) \quad 1.7$$

$$v' = g(v,\phi) \quad 1.15$$

$$t = f'(v',t',\phi') \quad 1.8$$

$$v = g'(v',\phi') \quad 1.20$$

Then we have that the distance is a function of two variables, the time a function of three variables and the velocity a function of two variables.

From the definition of moment 4.1 and energy 4.6 we have:

$$\vec{p} = \frac{E}{c^2}\vec{u} \quad 16.1$$

The elevated to the power of two supplies:

$$\frac{u^2}{c^2} = \frac{c^2}{E^2}p^2 \quad 16.2$$

Elevating to the power of two the energy formula we have:

$$E^2=\left(\frac{m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right)^2 \Rightarrow E^2-E^2\frac{u^2}{c^2}=m_0^2c^4$$

Where applying 16.2 we have:

$$E^2-E^2\frac{u^2}{c^2}=m_0^2c^4 \Rightarrow E^2-E^2\frac{c^2}{E^2}p^2=m_0^2c^4 \Rightarrow E=c\sqrt{p^2+m_0^2c^2} \qquad 4.8$$

From where we conclude that if the mass in resting position of a particle is null $m_0=zero$ the particle energy is equal to $E=cp$. 16.3

That applied in 16.2 supplies:

$$\frac{u^2}{c^2}=\frac{c^2}{E^2}p^2 \Rightarrow \frac{u^2}{c^2}=\frac{c^2}{(cp)^2}p^2 \Rightarrow u=c \qquad 16.4$$

From where we conclude that the movement of a particle with a null mass in resting position $m_0=zero$ will always be at the velocity of light $u=c$.

Applying in $E=cp$ the relations $E=yh$ and $c=y\lambda$ we have:

$$yh=y\lambda p \Rightarrow p=\frac{h}{\lambda} \text{ and in the same way } p'=\frac{h}{\lambda'} \qquad 16.5$$

Equation that relates the moment of a particle with a null mass in resting position with its own way length.

Elevating to the power of two the formula of moment transformation (4.9) we have:

$$\vec{p}'=\vec{p}-\frac{E}{c^2}\vec{v} \Rightarrow p'^2=p^2+\frac{E^2}{c^4}v^2-2\frac{E}{c^2}vp_x$$

Where applying $E=cp$ and $px=p\cos\phi=p\frac{ux}{c}$ we find:

$$p'^2=p^2+\frac{(cp)^2}{c^4}v^2-2\frac{cp}{c^2}vp\frac{ux}{c} \Rightarrow p'=p\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}} \Rightarrow p'=p\sqrt{K} \qquad 16.6$$

Where applying 16.5 results in:

$$p'=p\sqrt{K} \Rightarrow \frac{h}{\lambda'}=\frac{h}{\lambda}\sqrt{K} \Rightarrow \lambda'=\frac{\lambda}{\sqrt{K}} \text{ or inverted } \lambda=\frac{\lambda'}{\sqrt{K'}} \qquad 2.21$$

Where applying $c=y\lambda$ and $c=y'\lambda'$ we have:

$$y'=y\sqrt{K} \text{ or inverted } y=y'\sqrt{K'} \qquad 2.22$$

In § 2 we have the equations 2.21 and 2.22 applying the principle of relativity to the wave phase.

17 Transformation of H. Lorentz

For two observers in a relative movement, the equation that represents the principle of constancy of light speed for a random point A is:

$$x'^2+y'^2+z'^2-c^2t'^2=x^2+y^2+z^2-c^2t^2 \qquad 17.01$$

In this equation canceling the symmetric terms we have:
Nesta cancelando os termos simétricos obtemos:

$$x'^2-c^2t'^2=x^2-c^2t^2 \qquad 17.02$$

That we can write as:

$$(x'-ct')(x'+ct')=(x-ct)(x+ct) \qquad 17.03$$

If in this equation we define the proportion factors η and μ as:

$$\begin{cases}(x'-ct')=\eta(x-ct) & A\\(x'+ct')=\mu(x+ct) & B\end{cases} \qquad 17.04$$

where we must have $\eta.\mu=1$ to comply 17.03.

The equations 17.04 where first gotten by Albert Einstein.

When a ray of light moves in the plane y'z' to the observer O' we have x' = zero and x = vt and such conditions applied to the equation 17.02 supplies:

$$0-c^2t'^2=(vt)^2-c^2t^2 \Rightarrow t'=t\sqrt{1-\frac{v^2}{c^2}} \qquad 17.05$$

This result will also be supplied by the equations A and B of the group 17.04 under the same conditions:

$$\begin{cases}\left(0-ct\sqrt{1-\frac{v^2}{c^2}}\right)=\eta(vt-ct) & A\\\left(0+ct\sqrt{1-\frac{v^2}{c^2}}\right)=\mu(vt+ct) & B\end{cases} \qquad 17.06$$

From those we have:

$$\eta=\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \text{ and } \mu=\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \qquad 17.07$$

Where we have proven that $\eta.\mu=1$.

From the group 17.04 we have the Transformations of H. Lorentz:

$$x'=\frac{(\eta+\mu)}{2}x+\frac{(\mu-\eta)}{2}ct \qquad 17.08$$

$$ct'=\frac{(\mu-\eta)}{2}x+\frac{(\eta+\mu)}{2}ct \qquad 17.09$$

$$x=\frac{(\eta+\mu)}{2}x'+\frac{(\eta-\mu)}{2}ct' \qquad 17.10$$

$$ct=\frac{(\eta-\mu)}{2}x'+\frac{(\eta+\mu)}{2}ct' \qquad 17.11$$

Indexes equations $\frac{\eta+\mu}{2}$, $\frac{\mu-\eta}{2}$ and $\frac{\eta-\mu}{2}$:

$$\eta+\mu=\sqrt{\frac{1+\frac{V}{C}}{1-\frac{V}{C}}}+\sqrt{\frac{1-\frac{V}{C}}{1+\frac{V}{C}}}=\frac{1+\frac{V}{C}+1-\frac{V}{C}}{\sqrt{1-\frac{V}{C}}\sqrt{1+\frac{V}{C}}}=\frac{2}{\sqrt{1-\frac{V^2}{C^2}}}\Rightarrow\frac{\eta+\mu}{2}=\frac{1}{\sqrt{1-\frac{V^2}{C^2}}} \quad 17.12$$

$$\mu-\eta=\sqrt{\frac{1-\frac{V}{C}}{1+\frac{V}{C}}}-\sqrt{\frac{1+\frac{V}{C}}{1-\frac{V}{C}}}=\frac{1-\frac{V}{C}-1-\frac{V}{C}}{\sqrt{1+\frac{V}{C}}\sqrt{1-\frac{V}{C}}}=\frac{-2\frac{V}{C}}{\sqrt{1-\frac{V^2}{C^2}}}\Rightarrow\frac{\mu-\eta}{2}=\frac{-\frac{V}{C}}{\sqrt{1-\frac{V^2}{C^2}}} \quad 17.13$$

$$\eta-\mu=\sqrt{\frac{1+\frac{V}{C}}{1-\frac{V}{C}}}-\sqrt{\frac{1-\frac{V}{C}}{1+\frac{V}{C}}}=\frac{1+\frac{V}{C}-1+\frac{V}{C}}{\sqrt{1-\frac{V}{C}}\sqrt{1+\frac{V}{C}}}=\frac{2\frac{V}{C}}{\sqrt{1-\frac{V^2}{C^2}}}\Rightarrow\frac{\eta-\mu}{2}=\frac{\frac{V}{C}}{\sqrt{1-\frac{V^2}{C^2}}} \quad 17.14$$

Sagnac effect

When both observers' origins are equal the time is zeroed (t = t' = zero) in both referentials and two rays of light are emitted from the common origin, one in the positive direction (clockwise index c) of the axis x and x' with a wave front A_c and another in the negative direction (counter-clockwise index u) of the axis x and x' with a wave front A_u.

The propagation conditions above applied to the Lorentz equations supply the tables A and B below:

Table A

Equation	Clockwise ray (c)	Equation	Counter-clockwise ray (u)	Sum of rays
	Result		Result	
Condition	$x_c=ct_c$	Condition	$x_u=-ct_u$	
17.08	$x'_c=\mu ct_c$	17.08	$x'_u=-\eta ct_u$	
	$x'_c=\mu x_c$		$x'_u=\eta x_u$	$x'_c+x'_u=\mu x_c+\eta x_u$
17.09	$ct'_c=\mu ct_c$	17.09	$ct'_u=\eta ct_u$	$ct'_c+ct'_u=\mu ct_c+\eta ct_u$
	$x'_c=ct'_c$		$x'_u=-ct'_u$	

Table B

Equation	Clockwise ray (c)	Equation	Counter-clockwise ray (u)	Sum of rays
	Result		Result	
Condition	$x'_c=ct'_c$	Condition	$x'_u=-ct'_u$	
17.10	$x_c=\eta ct'_c$	17.10	$x_u=-\mu ct'_u$	
	$x_c=\eta x'_c$		$x_u=\mu x'_u$	$x_c+x_u=\eta x'_c+\mu x'_u$
17.11	$ct_c=\eta ct'_c$	17.11	$ct_u=\mu ct'_u$	$ct_c+ct_u=\eta ct'_c+\mu ct'_u$
	$x_c=ct_c$		$x_u=-ct_u$	

We observe that the tables A and B are inverse one to another.

When we form the group of the sum equations of the two rays from tables A and B:

$$\begin{cases} D'=ct'_c+ct'_u=\mu ct_c+\eta ct_u & A \\ D=ct_c+ct_u=\eta ct'_c+\mu ct'_u & B \end{cases} \quad 17.15$$

Where to the observer O' $D'=A_u \leftrightarrow A_c$ is the distance between the front waves A_u and A_c and where to the observer O $D=A_u \leftrightarrow A_c$ is the distance between the front waves A_u and A_c.

In the equations 17.15 above, due to the isotropy of space and time and the front waves $A_u \leftrightarrow A_c$ of the two rays of light being the same for both observers, the sum of rays of light e times must be invariable between the observers, which we can express by:

$$D'=D \Rightarrow ct'_c+ct'_u=ct_c+ct_u \Rightarrow \sum t'=\sum t \qquad 17.16$$

This result that generates an equation of isotropy of space and time can be called as the conservation of space and time principle.

The three hypothesis of propagation defined as follows will be applied in 17.15 and tested to prove the conservation of space and time principle given by 17.16:

Hypothesis A:

If the space and time are isotropic and there is no movement with no privilege of one observer considered over the other in an empty space then the propagation geometry of rays of light can be given by:

$|ct_c|=|ct'_u|$ and $|ct_u|=|ct'_c|$ 17.17

This hypothesis applied to the equation A or B of the group 17.15 complies to the space and time conservation principle given by 17.16.

The hypothesis 17.17 applied to the tables A and B results in:

$$Quadro\,A \begin{cases} ct'_c=\mu ct'_u & A \\ ct'_u=\eta ct'_c & B \end{cases} \qquad Quadro\,B \begin{cases} ct_c=\eta ct_u & C \\ ct_u=\mu ct_c & D \end{cases} \qquad 17.18$$

Hypothesis B:

If the space and time are isotropic but the observer O is in an absolute resting position in an empty space then the geometry of propagation of the rays of light is given by:

$$|ct_c|=|ct_u|=|ct| \qquad 17.19$$

That applied to the table A and B results in:

$$Quadro\,A \begin{cases} ct'_c=\mu ct & A \\ ct'_u=\eta ct & B \end{cases} \qquad Quadro\,B \begin{cases} ct=\eta ct'_c & C \\ ct=\mu ct'_u & D \end{cases} \qquad 17.20$$

$$\begin{cases} ct'_c=\mu^2 ct'_u & A \\ ct'_u=\eta^2 ct'_c & B \end{cases} \qquad 17.21$$

Summing A and B in 17.20 we have:

$$ct'_c+ct'_u=2ct\left(\frac{\eta+\mu}{2}\right) \Rightarrow D'=D\left(\frac{\eta+\mu}{2}\right) \Rightarrow D'=\frac{D}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \sum t'=\frac{\sum t}{\sqrt{1-\frac{v^2}{c^2}}} \qquad 17.22$$

This result doesn't comply with the conservation of space and time principle given by 17.16 and as $D' \neq D$ it results in a situation of four rays of light, two to each observer, and each ray of light with its respective independent front wave from the others.

Hypothesis C:

If the space and time are isotropic but the observer O' is in an absolute resting position in an empty space then the propagation geometry of the rays of light is given:

$$|ct'_c| = |ct'_u| = |ct'| \qquad 17.23$$

That applied to the tables A and B results in:

$$QuadroA \begin{cases} ct' = \mu ct_c & A \\ ct' = \eta ct_u & B \end{cases}$$
$$QuadroB \begin{cases} ct_c = \eta ct' & C \\ ct_u = \mu ct' & D \end{cases} \qquad 17.24$$

$$\begin{cases} ct_c = \eta^2 ct_u & A \\ ct_u = \mu^2 ct_c & B \end{cases} \qquad 17.25$$

Summing C and D in 17.24 we have:

$$ct_c + ct_u = 2ct'\left(\frac{\eta+\mu}{2}\right) \Rightarrow D = D'\left(\frac{\eta+\mu}{2}\right) \Rightarrow D = \frac{D'}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \sum t = \frac{\sum t'}{\sqrt{1-\frac{v^2}{c^2}}} \qquad 17.26$$

This result doesn't comply with the conservation of space and time principle exactly the same way as hypothesis B given by 17.16 and as $D' \neq D$ $D' \neq D$ it results in a situation of four rays of light, two to each observer and each ray of light with its respective independent front wave from the others.

Conclusion

The hypothesis A, B and C are completely compatible with the demand of isotropy of space and time as we can conclude with the geometry of propagations.

The result of hypothesis A is contrary to the result of hypothesis B and C despite of the relative movement of the observers not changing the front wave A_u relatively to the front wave A_c because the front waves have independent movement one from the other and from the observers.

The hypothesis A applied in the transformations of H. Lorentz complies with the conservation of space and time principle given by 17.16 showing the compatibility with the transformations of H. Lorentz with the hypothesis A. The application of hypothesis B and C in the transformations of H. Lorentz supplies the space and time deformations given by 17.22 and 17.26 because the transformations of H. Lorentz are not compatible with the hypothesis B and C.

For us to obtain the Sagnac effect we must consider that the observer O' is in an absolute resting position, hypothesis C above and that the path of the rays of light be of $2\pi R$:

$$ct'_c = ct'_u = ct' = 2\pi R \qquad 17.27$$

For the observer O the Sagnac effect is given by the time difference between the clockwise ray of light and the counter-clock ray of light $\Delta t = t_c - t_u$ that can be obtained using 17.24 (C-D), 17.27 and 17.14:

$$\Delta t = t_c - t_u = t'(\eta - \mu) = \frac{2\pi R}{c}\left(\frac{2\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{4\pi R v}{c\sqrt{c^2 - v^2}} \qquad 17.28$$

§9 The Sagnac Effect (continuation)

The moment the origins are the same the time is zeroed (t = t' = zero) at both sides of the referential and the rays of light are emitted from the common origin, one in the positive way (clockwise index c) of the axis x and x' with a wave front A_c and the other one in the negative way (counter clockwise index u) of the axis x and x' with wave front A_u.

The projected ray of light in the positive way (clockwise index c) of the axis x and x' is equationed by $x_c = c t_c$ and $x'_c = c t'_c$ that applied to the Table I supplies:

$$c t'_c = c t_c\left(1 - \frac{v_c}{c}\right) \Rightarrow c t'_c = c t_c K_c \text{ (1.7)} \quad c t_c = c t'_c\left(1 + \frac{v'_c}{c}\right) \Rightarrow c t_c = c t'_c K'_c \text{ (1.8)} \qquad 9.11$$

$$v'_c = \frac{v_c}{\left(1 - \frac{v_c}{c}\right)} \Rightarrow v'_c = \frac{v_c}{K_c} \text{ (1.15)} \qquad v_c = \frac{v'_c}{\left(1 + \frac{v'_c}{c}\right)} \Rightarrow v_c = \frac{v'_c}{K'_c} \text{ (1.20)} \qquad 9.12$$

From those we deduct that the distance between the observers is given by:

$$d_c = v_c t_c = v'_c t'_c \qquad 9.13$$

Where we have:

$$\left(1 - \frac{v_c}{c}\right)\left(1 + \frac{v'_c}{c}\right) = K_c K'_c = 1 \qquad 9.14$$

The ray of light project in the negative way (counter clockwise index u) of the axis x and x' is equationed by $x_u = -c t_u$ and $x'_u = -c t'_u$: that applied to the Table I gives:

$$c t'_u = c t_u\left(1 + \frac{v_u}{c}\right) \Rightarrow c t'_u = c t_u K_u \text{ (1.7)} \qquad c t_u = c t'_u\left(1 - \frac{v'_u}{c}\right) \Rightarrow c t_u = c t'_u K'_u \text{ (1.8)} \qquad 9.15$$

$$v'_u = \frac{v_u}{\left(1 + \frac{v_u}{c}\right)} \Rightarrow v'_u = \frac{v_u}{K_u} \text{ (1.15)} \qquad v_u = \frac{v'_u}{\left(1 - \frac{v'_u}{c}\right)} \Rightarrow v_u = \frac{v'_u}{K'_u} \text{ (1.20)} \qquad 9.16$$

From those we deduct that the distance between the observers is given by:

$$d_u = v_u t_u = v'_u t'_u \qquad 9.17$$

Where we have:

$$\left(1 + \frac{v_u}{c}\right)\left(1 - \frac{v'_u}{c}\right) = K_u K'_u = 1 \qquad 9.18$$

We must observe that at first there is no relationship between the equations 9.11 to 9.14 with the equations 9.15 to 9.18.

With the propagation conditions described we form the following Tables A and B:

Table A

Equation	Clockwise ray of light (c)	Equation	Counter clockwise ray of light (u)	Sum of the rays of light
	Result		Result	
Condition	$x_c = c t_c$	Condition	$x_u = -c t_u$	
1.2	$x'_c = c t_c K_c$	1.2	$x'_u = -c t_u K_u$	

	$x'_c=x_cK_c$		$x'_u=x_uK_u$	$x'_c+x'_u=x_cK_c+x_uK_u$
1.7	$ct'_c=ct_cK_c$	1.7	$ct'_u=ct_uK_u$	$ct'_c+ct'_u=ct_cK_c+ct_uK_u$
	$x'_c=ct'_c$		$x'_u=-ct'_u$	

Table B

Equation	Clockwise ray of light (c)	Equation	Counter clockwise ray of light (u)	Sum of the rays of light
	Result		Result	
Condition	$x'_c=ct'_c$	Condition	$x'_u=-ct'_u$	
1.4	$x_c=ct'_cK'_c$	1.4	$x_u=-ct'_uK'_u$	
	$x_c=x'_cK'_c$		$x_u=x'_uK'_u$	$x_c+x_u=x'_cK'_c+x'_uK'_u$
1.8	$ct_c=ct'_cK'_c$	1.8	$ct_u=ct'_uK'_u$	$ct_c+ct_u=ct'_cK'_c+ct'_uK'_u$
	$x_c=ct_c$		$x_u=-ct_u$	

We observe that for the rays of light with the same direction the Tables A and B are inverse from each other.

Forming the equations group of the sum of the rays of light of the Tables A and B:

$$\begin{cases} D'=ct'_c+ct'_u=ct_cK_c+ct_uK_u & A \\ D=ct_c+ct_u=ct'_cK'_c+ct'_uK'_u & B \end{cases} \qquad 9.19$$

Where for the observer O' $D'=A_u\leftrightarrow A_c$ is the distance between the wave fronts A_u and A_c and where for the observer O $D=A_u\leftrightarrow A_c$ is the distance between the wave fronts A_u and A_c.

In the equations above 9.19 due to the isotropy of the space and time and the wave fronts $A_u\leftrightarrow A_c$ of the rays of light being the same for both observers, the sumo of the rays of light and of times must be invariable between the observers, which is expressed by:

$$D'=D\Rightarrow ct'_c+ct'_u=ct_c+ct_u\Rightarrow\sum t'=\sum t \qquad 9.20$$

This result that equations the isotropy of space and time can be called as the space and time conservation principle.

The three hypothesis of propagations defined next will be applied in 9.19 and tested to prove the compliance of the conservation of space and time principle given by 9.20. With these hypotheses we create a bond between the equations 9.11 to 9.14 with the equations 9.15 to 9.18.

Hypothesis A:

If the space and time are isotropic and there is movement with any privilege of any observer over each other in the empty space then the propagation geometry of the rays of light is equationed by:

$$\begin{cases} ct_c=ct'_u\Rightarrow t_c=t'_u\Rightarrow v_c=v'_u\Rightarrow K_c=K'_u & A \\ ct_u=ct'_c\Rightarrow t_u=t'_c\Rightarrow v_u=v'_c\Rightarrow K_u=K'_c & B \end{cases} \qquad 9.21$$

With those we deduct that the distance between the observers is given by:

$$d_c=d_u=v_ct_c=v'_ct'_c=v_ut_u=v'_ut'_u \qquad 9.22$$

Results that applied in the equations A or B of the group 9.19 complies with the conservation of space and time principle given by 9.20, showing that the Doppler effect in the clockwise and counter clockwise rays of light are compensated in the referentials.

Hypothesis B:

If the space and time are isotropic but the observer O is in an absolute resting position in the empty space then the propagation geometry of the rays of light is equationed by:

$$\begin{cases} ct_c = ct_u = ct & A \\ v_c = v_u = v & B \\ v_c t_c = v_u t_u = vt & C \end{cases} \qquad 9.23$$

With those we deduct that the distance between the observers is given by:

$$d_c = d_u = vt = v'_c t'_c = v'_u t'_u \qquad 9.24$$

Results that applied in the equations A or B of the group 9.19 complies with the conservation of space and time principle given by 9.20, showing that the Doppler effect in the clockwise and counter clockwise rays of light are compensated in the referentials.

Hypothesis C:

If the space and time are isotropic but the observer O is in an absolute resting position in the empty space then the propagation geometry of the rays of light is equationed by:

$$\begin{cases} ct'_c = ct'_u = ct' & A \\ v'_c = v'_u = v' & B \\ v'_c t'_c = v'_u t'_u = v't' & C \end{cases} \qquad 9.25$$

With those we deduct that the distance between the observers is given by:

$$d_c = d_u = v't' = v_c t_c = v_u t_u \qquad 9.26$$

Results that applied in the equations A or B of the group 9.19 complies with the conservation of space and time principle given by 9.20, showing that the Doppler effect in the clockwise and counter clockwise rays of light are compensated in the referentials.

In order to obtain the Sagnac effect we consider that the observer O' is in an absolute resting position, hypothesis C above and that the rays of light course must be of $2\pi R$:

$$ct'_c = ct'_u = ct' = 2\pi R \qquad 9.27$$

Applying the hypothesis C in 9.11 and 9.15 we have:

$$t_c = t'_c K'_c \Rightarrow t_c = t'\left(1 + \frac{v'}{c}\right) \qquad 9.28$$

$$t_u = t'_u K'_u \Rightarrow t_u = t'\left(1 - \frac{v'}{c}\right) \qquad 9.29$$

For the observer O the Sagnac effect is given by the time difference between course of the clockwise ray of light and the counter clock ray of $\Delta t = t_c - t_u$ that can be obtained making (9.28 – 9.29) and applying 9.27 making:

$$\Delta t = t_c - t_u = t'\left(1 + \frac{v'}{c}\right) - t'\left(1 - \frac{v'}{c}\right) = \frac{2v't'}{c} = \frac{4\pi R v'}{c^2} \qquad 9.30$$

The equation $\Delta t = \frac{2v't'}{c} = \frac{2v_c t_c}{c} = \frac{2v_u t_u}{c}$ is exactly the result obtained from the geometry analysis of the propagation of the clockwise and counter clockwise rays of light in a circumference showing the coherence of the hypothesis adopted by the Undulating Relativity.

In 9.30 applying 9.12 and 9.16 we have the final result due to V_C and V_u:

$$\Delta t = t_c - t_u = \frac{2Vt'}{c} = \frac{4\pi R V'}{c^2} = \frac{4\pi R V_c}{c^2 - cV_c} = \frac{4\pi R V_u}{c^2 + cV_u}$$ 9.31

The classic formula of the Sagnac effect is given as:

$$\Delta t = t_c - t_u = \frac{4\pi R V}{c^2 - V^2}$$ 9.32

From the propagation geometry we have:

$$\Delta t = \frac{2Vt}{c}$$ 9.33

The classic times would be given by:

$$t = \frac{2\pi R}{c}$$ 9.34

$$t_c = \frac{2\pi R}{c - V}$$ 9.35

$$t_u = \frac{2\pi R}{c + V}$$ 9.36

Applying 9.34, 9.35 and 9.36 in 9.33 we have:

$$\Delta t = \frac{2V}{c}\frac{2\pi R}{c} = \frac{4\pi R V}{c^2}$$ 9.37

$$\Delta t_c = \frac{2V}{c}\frac{2\pi R}{(c - V)} = \frac{4\pi R V}{c^2 - cV}$$ 9.38

$$\Delta t_u = \frac{2V}{c}\frac{2\pi R}{(c + V)} = \frac{4\pi R V}{c^2 + cV}$$ 9.39

The results 9.37, 9.38 and 9.39 are completely different from 9.32.

§18 The Michelson & Morley experience

The traditional analysis that supplies the solution for the null result of this experience considers a device in a resting position at the referential of the observer O’ that emits two rays of light, one horizontal in the x’ direction (clockwise index c) and another vertical in the direction y’. The horizontal ray of light (clockwise index c) runs until a mirror placed in x’ = L at this point the ray of light reflects (counter clockwise index u) and returns to the origin of the referential where x’ = zero. The vertical ray of light runs until a mirror placed in y’ = L reflects and returns to the origin of the referential where y’ = zero.

In the traditional analysis according to the speed of light constancy principle for the observer O’ the rays of light track is given by:

$$ct'_c = ct'_u = L$$ 18.01

For the observer O’ the sum of times of the track of both rays of light along the x’ axis is:

$$\sum t'_{x'} = t'_c + t'_u = \frac{L}{c} + \frac{L}{c} = \frac{2L}{c}$$ 18.02

In the traditional analysis for the observer O' the sum of times of the track of both rays of light along the y' axis is:

$$\sum t'_{y'} = t'_{+} + t'_{-} = \frac{L}{c} + \frac{L}{c} = \frac{2L}{c} \qquad 18.03$$

As we have $\sum t'_{x'} = \sum t'_{y'} = \frac{2L}{c}$ there is no interference fringe and it is applied the null result of the Michelson & Morley experience.

In this traditional analysis the identical track of the clockwise and counter clockwise rays of light in the equation 18.01 that originates the null result of the Michelson & Morley experience contradicts the Sagnac effect that is exactly the time difference existing between the track of the clockwise and counter clockwise rays of light.

Based on the Undulating Relativity we make a deeper analysis of the Michelson & Morley experience obtaining a result that complies completely with the Sagnac effect.

Observing that the equation 18.01 corresponds to the hypothesis C of the paragraph §9.
Applying 18.01 in 9.19 we have:

$$\begin{cases} D' = ct'_c + ct'_u = ct_c K_c + ct_u K_u \Rightarrow D' = L + L = ct_c K_c + ct_u K_u & A \\ D = ct_c + ct_u = ct'_c K'_c + ct'_u K'_u \Rightarrow D = ct_c + ct_u = LK'_c + LK'_u = L\left(K'_c + K'_u\right) & B \end{cases} \qquad 18.04$$

From 18.04 A we have:

$$D' = 2L = ct_c\left(1 - \frac{v_c}{c}\right) + ct_u\left(1 + \frac{v_u}{c}\right) \Rightarrow D' = 2L = ct_c - v_c t_c + ct_u + v_u t_u \qquad 18.05$$

Where applying 9.26 we have:

$$D' = 2L = ct_c + ct_u \Rightarrow \sum t_x = t_c + t_u = \frac{2L}{c} \qquad 18.06$$

In 18.04 B we have:

$$D = ct_c + ct_u = L\left[\left(1 + \frac{v'_c}{c}\right) + \left(1 - \frac{v'_u}{c}\right)\right] \qquad 18.07$$

Where applying 9.25 B we have:

$$D = ct_c + ct_u = 2L \Rightarrow \sum t_x = t_c + t_u = \frac{2L}{c} \qquad 18.08$$

The equations 18.06 and 18.08 demonstrate that the Doppler effect in the clockwise and counter clockwise rays of light compensate itself in the referential of the observer O resulting in:

$$\sum t'_{y'} = \sum t'_{x'} = \sum t_x = \frac{2L}{c} \qquad 18.09$$

Because of this, according to the Undulating Relativity in the Michelson & Morley experience we can predict that the clockwise ray of light has a different track from the counter clockwise ray of light according to the formula 18.08 obtaining also the null result for the experience and matching then with the Sagnac effect. This supposition cannot be made based on the Einstein's Special Relativity because according to 17.26 we have:

$$\sum t'_{x'} \neq \sum t_x \qquad 18.10$$

§19 Regression of the perihelion of Mercury of 7,13"

Let us imagine the Sun located in the focus of an ellipse that coincides with the origin of a system of coordinates (x,y,z) with no movement in relation to denominated fixed stars and that the planet Mercury is in a movement governed by the force of gravitational attraction with the Sun describing an elliptic orbit in the plan (x,y) according to the laws of Kepler and the formula of the Newton's gravitational attraction law:

$$\vec{F}=\frac{-GMm_o}{r^2}\hat{r}=\frac{-(6{,}67.10^{-11})(1{,}98.10^{30})(3{,}28.10^{23})}{r^2}\hat{r}=\frac{-k}{r^2}\hat{r} \qquad 19.01$$

The sub index "o" indicating mass in relative rest to the observer.

To describe the movement we will use the known formulas:

$$\vec{r}=r\hat{r} \qquad 19.02$$

$$\vec{u}=\frac{d\vec{r}}{dt}=\frac{d(r\hat{r})}{dt}=\frac{dr}{dt}\hat{r}+r\frac{d\phi}{dt}\hat{\phi} \qquad 19.03$$

$$u^2=\vec{u}\vec{u}=\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\phi}{dt}\right)^2 \qquad 19.04$$

$$\vec{a}=\frac{d\vec{u}}{dt}=\frac{d^2\vec{r}}{dt^2}=\frac{d^2(r\hat{r})}{dt^2}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]\hat{r}+\left[2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right]\hat{\phi} \qquad 19.05$$

The formula of the relativity force is given by:

$$\vec{F}=\frac{d}{dt}\left(\frac{m_o\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right)=\frac{m_o}{\sqrt{1-\frac{u^2}{c^2}}}\vec{a}+\frac{m_o}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\frac{u}{c^2}\frac{du}{dt}\vec{u}=\frac{m_o}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}\left[\left(1-\frac{u^2}{c^2}\right)\vec{a}+\left(\vec{u}\frac{d\vec{u}}{dt}\right)\frac{\vec{u}}{c^2}\right] \qquad 19.06$$

In this the first term corresponds to the variation of the mass with the speed and the second as we will see later in 19.22 corresponds to the variation of the energy with the time.

With this and the previous formulas we obtain:

$$\vec{F}=\frac{m_o}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}\left\{\begin{array}{l}\left(1-\frac{u^2}{c^2}\right)\left\{\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]\hat{r}+\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\hat{\phi}\right\}+\\+\left\{\frac{dr}{dt}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]+r\frac{d\phi}{dt}\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\right\}\frac{1}{c^2}\left(\frac{dr}{dt}\hat{r}+r\frac{d\phi}{dt}\hat{\phi}\right)\end{array}\right\} \qquad 19.07$$

$$\vec{F}=\frac{m_o}{\left(1-u^2/c^2\right)^{3/2}}\left\{\begin{array}{l}\left\{\left(1-\frac{u^2}{c^2}\right)\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]+\left\{\frac{dr}{dt}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]+r\frac{d\phi}{dt}\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\right\}\frac{1}{c^2}\frac{dr}{dt}\right\}\hat{r}+\\+\left\{\left(1-\frac{u^2}{c^2}\right)\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)+\left\{\frac{dr}{dt}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]+r\frac{d\phi}{dt}\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\right\}\frac{r}{c^2}\frac{d\phi}{dt}\right\}\hat{\phi}\end{array}\right\} \qquad 19.08$$

In this we have the transverse and radial component given by:

$$\vec{F}_{\hat{r}}=\frac{m_o}{\left(1-u^2/c^2\right)^{3/2}}\left\{\left(1-\frac{u^2}{c^2}\right)\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]+\left\{\frac{dr}{dt}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]+r\frac{d\phi}{dt}\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\right\}\frac{1}{c^2}\frac{dr}{dt}\right\}\hat{r} \quad 19.09$$

$$\vec{F}_{\hat{\phi}}=\frac{m_o}{\left(1-u^2/c^2\right)^{3/2}}\left\{\left(1-\frac{u^2}{c^2}\right)\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)+\left\{\frac{dr}{dt}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]+r\frac{d\phi}{dt}\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\right\}\frac{r}{c^2}\frac{d\phi}{dt}\right\}\hat{\phi} \quad 19.10$$

As the gravitational force is central we should have to null the traverse component $\vec{F}_{\hat{\phi}}=zero$ so we have:

$$\vec{F}_{\hat{\phi}}=\frac{m_o}{\left(1-u^2/c^2\right)^{3/2}}\left\{\left(1-\frac{u^2}{c^2}\right)\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)+\left\{\frac{dr}{dt}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]+r\frac{d\phi}{dt}\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\right\}\frac{r}{c^2}\frac{d\phi}{dt}\right\}\hat{\phi}=zero \quad 19.11$$

From where we have:

$$\frac{\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)}{\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]}=\frac{\frac{-r}{c^2}\frac{dr}{dt}\frac{d\phi}{dt}}{\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right]} \qquad \frac{\left(2r\frac{dr}{dt}\frac{d\phi}{dt}+r^2\frac{d^2\phi}{dt^2}\right)}{r^2\frac{d\phi}{dt}}=\frac{\frac{-1}{c^2}\frac{dr}{dt}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]}{\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right]} \quad 19.12$$

From the radial component $\vec{F}_{\hat{r}}$ we have:

$$\vec{F}_{\hat{r}}=\frac{m_o}{\left(1-u^2/c^2\right)^{3/2}}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]\left\{\left(1-\frac{u^2}{c^2}\right)+\left\{\frac{dr}{dt}+\frac{r\frac{d\phi}{dt}\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)}{\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]}\right\}\frac{1}{c^2}\frac{dr}{dt}\right\}\hat{r} \quad 19.13$$

That applying 19.12 we have:

$$\vec{F}_{\hat{r}}=\frac{m_o}{\left(1-u^2/c^2\right)^{3/2}}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]\left\{\left(1-\frac{u^2}{c^2}\right)+\left\{\frac{dr}{dt}-\frac{r\frac{d\phi}{dt}\left(\frac{r}{c^2}\frac{dr}{dt}\frac{d\phi}{dt}\right)}{\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right]}\right\}\frac{1}{c^2}\frac{dr}{dt}\right\}\hat{r} \quad 19.14$$

That simplifying results in:

$$\vec{F}_{\hat{r}}=\frac{m_o}{\sqrt{1-\frac{u^2}{c^2}}}\frac{\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]}{\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right]}\hat{r} \quad 19.15$$

This equaled to Newton's gravitational force results in the relativistic gravitational force:

$$\vec{F}_{\hat{r}}=\frac{m_o}{\sqrt{1-\frac{u^2}{c^2}}}\frac{\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]}{\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right]}\hat{r}=\frac{-GM_om_o}{r^2}\hat{r}=\frac{-k}{r^2}\hat{r} \quad 19.16$$

As the gravitational force is central it should assist the theory of conservation of the energy (E) that is written as:

$$E = E_k + E_p = \text{constant}. \qquad 19.17$$

Where the kinetic energy (E_k) is given by:

$$E_k = mc^2 - m_o c^2 = m_o c^2 \left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right) \qquad 19.18$$

And the potential energy (E_p) gravitational by:

$$E_p = \frac{-GM_o m_o}{r} = \frac{-k}{r} \qquad 19.19$$

Resulting in:

$$E = m_o c^2 \left[\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right] - \frac{k}{r} = \text{Constant}. \qquad 19.20$$

As the total energy (E) it is constant we should have:

$$\frac{dE}{dt} = \frac{dE_k}{dt} + \frac{dE_p}{dt} = zero. \qquad 19.21$$

Then we have:

$$\frac{dE_k}{dt} = \frac{m_o u}{\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}} \frac{du}{dt} \qquad 19.22$$

$$\frac{dE_p}{dt} = \frac{k}{r^2} \frac{dr}{dt} \qquad 19.23$$

Resulting in:

$$\frac{dE}{dt} = \frac{dE_k}{dt} + \frac{dE_p}{dt} = zero \Rightarrow \frac{m_o u}{\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}} \frac{du}{dt} + \frac{k}{r^2} \frac{dr}{dt} = zero \Rightarrow \frac{m_o u}{\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}} \frac{du}{dt} = \frac{-k}{r^2} \frac{dr}{dt} \qquad 19.24$$

This applied in the relativistic force 19.06 and equaled to the gravitational force 19.01 results in:

$$\vec{F} = \frac{m_o}{\sqrt{1 - \frac{u^2}{c^2}}} \vec{a} - \frac{1}{c^2} \frac{k}{r^2} \frac{dr}{dt} \vec{u} = \frac{-k}{r^2} \hat{r} \qquad 19.25$$

In this substituting the previous variables we get:

$$\vec{F}=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\left\{\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]\hat{r}+\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\hat{\phi}\right\}-\frac{1}{c^2}\frac{k}{r^2}\frac{dr}{dt}\left(\frac{dr}{dt}\hat{r}+r\frac{d\phi}{dt}\hat{\phi}\right)=\frac{-k}{r^2}\hat{r} \quad 19.26$$

From this we obtain the radial component $\vec{F}_{\hat{r}}$ equals to:

$$\vec{F}_{\hat{r}}=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]-\frac{1}{c^2}\frac{k}{r^2}\left(\frac{dr}{dt}\right)^2=\frac{-k}{r^2} \quad 19.27$$

That easily becomes the relativistic gravitational force 19.16.

From 19.26 we obtain the traverse component $\vec{F}_{\hat{\phi}}$ equals to:

$$\vec{F}_{\hat{\phi}}=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)-\frac{1}{c^2}\frac{k}{r}\frac{dr}{dt}\frac{d\phi}{dt}=zero \quad 19.28$$

From this last one we have:

$$\frac{2r\frac{dr}{dt}\frac{d\phi}{dt}+r^2\frac{d^2\phi}{dt^2}}{r^2\frac{d\phi}{dt}}=\frac{1}{m_0}\frac{k}{c^2r^2}\frac{dr}{dt}\sqrt{1-\frac{u^2}{c^2}} \quad 19.29$$

As the gravitational force is central it should also assist the theory of conservation of the angular moment that is written as:

$\vec{L}=\vec{r}\times\vec{p}=$constant. 19.30

$$\vec{L}=\vec{r}\times\vec{p}=\vec{r}\times\frac{m_0\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}=r\hat{r}\times\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\left(\frac{dr}{dt}\hat{r}+r\frac{d\phi}{dt}\hat{\phi}\right)=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}r^2\frac{d\phi}{dt}\left(\hat{r}\times\hat{\phi}\right)=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}r^2\frac{d\phi}{dt}\hat{k} \quad 19.31$$

$\vec{L}=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}r^2\frac{d\phi}{dt}\hat{k}=L\hat{k}$ = constant. 19.32

$$\frac{d\vec{L}}{dt}=\frac{d\left(L\hat{k}\right)}{dt}=\frac{d\left(L\right)\hat{k}}{dt}+\frac{Ld\left(\hat{k}\right)}{dt}=\frac{d\left(L\right)\hat{k}}{dt}=zero\Rightarrow\frac{d\left(L\right)}{dt}=zero \quad 19.33$$

Resulting in L that is constant.

In 19.33 we had $\frac{d\hat{k}}{dt}=zero$ because the movement is in the plane (x,y).

Deriving L we find:

$$\frac{dL}{dt}=\frac{d}{dt}\left(\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}r^2\frac{d\phi}{dt}\right)=\frac{1}{c^2}\frac{m_0 u}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\frac{du}{dt}r^2\frac{d\phi}{dt}+\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\left(2r\frac{dr}{dt}\frac{d\phi}{dt}+r^2\frac{d^2\phi}{dt^2}\right)=zero \qquad 19.34$$

From that we have:

$$\frac{\left(2r\frac{dr}{dt}\frac{d\phi}{dt}+r^2\frac{d^2\phi}{dt^2}\right)}{r^2\frac{d\phi}{dt}}=\frac{-u}{\left(1-\frac{u^2}{c^2}\right)}\frac{du}{dt}\frac{1}{c^2} \qquad 19.35$$

Equaling 19.12 originating from the theory of the central force with 19.29 originating from the theory of conservation of the energy and 19.35 originating from the theory of conservation of the angular moment we have:

$$\frac{\left(2r\frac{dr}{dt}\frac{d\phi}{dt}+r^2\frac{d^2\phi}{dt^2}\right)}{r^2\frac{d\phi}{dt}}=\frac{\frac{-1}{c^2}\frac{dr}{dt}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]}{\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right]}=\frac{k}{m_0c^2r^2}\frac{dr}{dt}\sqrt{1-\frac{u^2}{c^2}}=\frac{-u}{\left(1-\frac{u^2}{c^2}\right)}\frac{du}{dt}\frac{1}{c^2} \qquad 19.36$$

From the last two equality we obtain 19.24 and from the two of the middle we obtain 19.16.

For solution of the differential equations we will use the same method used in the Newton's theory.

Let us assume $w=\frac{1}{r}$ 19.37

The differential total of this is $dw=\frac{\partial w}{\partial r}dr \Rightarrow dw=\frac{-1}{r^2}dr$ 19.38

From where we have $\frac{dw}{d\phi}=\frac{-1}{r^2}\frac{dr}{d\phi}$ e $\frac{dw}{dt}=\frac{-1}{r^2}\frac{dr}{dt}$ 19.39

From the module of the angular moment we have $\frac{d\phi}{dt}=\frac{L}{m_0r^2}\sqrt{1-\frac{u^2}{c^2}}$ 19.40

From where we have $\frac{dr}{dt}=\frac{L}{m_0r^2}\frac{dr}{d\phi}\sqrt{1-\frac{u^2}{c^2}}$ 19.41

Where applying 19.39 we have $\frac{dr}{dt}=\frac{-L}{m_0}\frac{dw}{d\phi}\sqrt{1-\frac{u^2}{c^2}}$ 19.42

That derived supplies $\frac{d^2r}{dt^2}=\frac{d\phi}{dt}\frac{dt}{d\phi}\frac{d}{dt}\left(\frac{-L}{m_0}\frac{dw}{d\phi}\sqrt{1-\frac{u^2}{c^2}}\right)$ 19.43

Where applying 19.40 and deriving we have:

$$\frac{d^2r}{dt^2}=\frac{L}{m_0r^2}\sqrt{1-\frac{u^2}{c^2}}\frac{d}{d\phi}\left(\frac{-L}{m_0}\frac{dw}{d\phi}\sqrt{1-\frac{u^2}{c^2}}\right)=\frac{-L^2}{m_0^2r^2}\sqrt{1-\frac{u^2}{c^2}}\left[\frac{d^2w}{d\phi^2}\sqrt{1-\frac{u^2}{c^2}}+\frac{dw}{d\phi}\frac{d}{d\phi}\left(\sqrt{1-\frac{u^2}{c^2}}\right)\right] \quad 19.44$$

In this with 19.36 the radical derived is obtained this way:

$$\frac{d}{dt}\left(\sqrt{1-\frac{u^2}{c^2}}\right)=\frac{-1}{\sqrt{1-u^2/c^2}}\frac{u}{c^2}\frac{du}{dt}=\frac{k}{m_0c^2r^2}\frac{dr}{dt}\left(1-\frac{u^2}{c^2}\right)=\frac{-k}{m_0c^2}\frac{dw}{dt}\left(1-\frac{u^2}{c^2}\right) \quad 19.45$$

$$\frac{d}{d\phi}\left(\sqrt{1-\frac{u^2}{c^2}}\right)=\frac{-1}{\sqrt{1-u^2/c^2}}\frac{u}{c^2}\frac{du}{d\phi}=\frac{k}{m_0c^2r^2}\frac{dr}{d\phi}\left(1-\frac{u^2}{c^2}\right)=\frac{-k}{m_0c^2}\frac{dw}{d\phi}\left(1-\frac{u^2}{c^2}\right) \quad 19.46$$

That applied in 19.44 supplies:

$$\frac{d^2r}{dt^2}=\frac{-L^2}{m_0^2r^2}\sqrt{1-\frac{u^2}{c^2}}\left[\frac{d^2w}{d\phi^2}\sqrt{1-\frac{u^2}{c^2}}-\frac{k}{m_0c^2}\left(\frac{dw}{d\phi}\right)^2\left(1-\frac{u^2}{c^2}\right)\right] \quad 19.47$$

Simplified results:

$$\frac{d^2r}{dt^2}=\frac{L^2k}{m_0^3c^2r^2}\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}\left(\frac{dw}{d\phi}\right)^2-\frac{L^2}{m_0^2r^2}\left(1-\frac{u^2}{c^2}\right)\frac{d^2w}{d\phi^2} \quad 19.48$$

Let us find the second derived of the angle deriving 19.40:

$$\frac{d^2\phi}{dt^2}=\frac{d}{dt}\left(\frac{L}{m_0r^2}\sqrt{1-\frac{u^2}{c^2}}\right)=\frac{-2L}{m_0r^3}\frac{dr}{dt}\sqrt{1-\frac{u^2}{c^2}}+\frac{L}{m_0r^2}\frac{d}{dt}\left(\sqrt{1-\frac{u^2}{c^2}}\right) \quad 19.49$$

In this applying 19.42 and 19.45 and simplifying we have:

$$\frac{d^2\phi}{dt^2}=\frac{2L^2}{m_0^2r^3}\frac{dw}{d\phi}\left(1-\frac{u^2}{c^2}\right)-\frac{L^2k}{m_0^3c^2r^4}\frac{dw}{d\phi}\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}} \quad 19.50$$

Applying in 19.04 the equations 19.40 and 19.42 and simplifying we have:

$$u^2=\frac{L^2}{m_0^2}\left(1-\frac{u^2}{c^2}\right)\left[\left(\frac{dw}{d\phi}\right)^2+\frac{1}{r^2}\right] \quad 19.51$$

The equation of the relativistic gravitational force 19.16 remodeled is:

$$\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2=\sqrt{1-\frac{u^2}{c^2}}\left[1-\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\right]\frac{-k}{m_0r^2} \quad 19.52$$

In this applying the formulas above we have:

$$\frac{L^2k}{m_0^3c^2r^2}\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}\left(\frac{dw}{d\phi}\right)^2-\frac{L^2}{m_0^2r^2}\left(1-\frac{u^2}{c^2}\right)\frac{d^2w}{d\phi^2}-r\left(\frac{L}{m_0r^2}\sqrt{1-\frac{u^2}{c^2}}\right)^2=\sqrt{1-\frac{u^2}{c^2}}\left[1-\frac{1}{c^2}\left(\frac{-L}{m_0}\frac{dw}{d\phi}\sqrt{1-\frac{u^2}{c^2}}\right)^2\right]\frac{-k}{m_0r^2}$$

$$\frac{L^2k}{m_0^3c^2r^2}\left(1-\frac{u^2}{c^2}\right)\left(\frac{dw}{d\phi}\right)^2-\frac{L^2}{m_0^2r^2}\sqrt{1-\frac{u^2}{c^2}}\frac{d^2w}{d\phi^2}-\frac{L^2}{m_0^2r^3}\sqrt{1-\frac{u^2}{c^2}}=\left[1-\frac{1}{c^2}\left(\frac{-L}{m_0}\frac{dw}{d\phi}\sqrt{1-\frac{u^2}{c^2}}\right)^2\right]\frac{-k}{m_0r^2}$$

$$\frac{L^2k}{m_0^3c^2r^2}\left(1-\frac{u^2}{c^2}\right)\left(\frac{dw}{d\phi}\right)^2-\frac{L^2}{m_0^2r^2}\sqrt{1-\frac{u^2}{c^2}}\frac{d^2w}{d\phi^2}-\frac{L^2}{m_0^2r^3}\sqrt{1-\frac{u^2}{c^2}}=\frac{-k}{m_0r^2}+\frac{L^2k}{m_0^3r^2}\frac{1}{c^2}\left(1-\frac{u^2}{c^2}\right)\left(\frac{dw}{d\phi}\right)^2$$

$$-\frac{L^2}{m_0^2r^2}\sqrt{1-\frac{u^2}{c^2}}\frac{d^2w}{d\phi^2}-\frac{L^2}{m_0^2r^3}\sqrt{1-\frac{u^2}{c^2}}=\frac{-k}{m_0r^2}$$

$$\frac{d^2w}{d\phi^2}+\frac{1}{r}=\frac{m_0k}{L^2\sqrt{1-\frac{u^2}{c^2}}}$$

$$\frac{d^2w}{d\phi^2}+\frac{1}{r}=\frac{m_0k}{\left(\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}r^2\frac{d\phi}{dt}\right)^2\sqrt{1-\frac{u^2}{c^2}}}$$

$$\frac{d^2w}{d\phi^2}+\frac{1}{r}=\frac{m_0k\sqrt{1-\frac{u^2}{c^2}}}{m_0^2r^4\left(\frac{d\phi}{dt}\right)^2}$$

$$\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)^2=\left[\frac{k\sqrt{1-\frac{u^2}{c^2}}}{m_0r^4\left(\frac{d\phi}{dt}\right)^2}\right]^2$$

$$\left(\frac{d^2w}{d\phi^2}\right)^2+\frac{2}{r}\frac{d^2w}{d\phi^2}+\frac{1}{r^2}=\frac{k^2\left(1-\frac{u^2}{c^2}\right)}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^4}$$

$$\left(\frac{d^2w}{d\phi^2}\right)^2+\frac{2}{r}\frac{d^2w}{d\phi^2}+\frac{1}{r^2}=\frac{k^2}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^4}-\frac{\frac{k^2}{c^2}u^2}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^4}$$

$$\left(\frac{d^2w}{d\phi^2}\right)^2+\frac{2}{r}\frac{d^2w}{d\phi^2}+\frac{1}{r^2}=\frac{k^2}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^4}-\frac{\frac{k^2}{c^2}\left[\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\phi}{dt}\right)^2\right]}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^4}$$

$$\left(\frac{d^2w}{d\phi^2}\right)^2+\frac{2}{r}\frac{d^2w}{d\phi^2}+\frac{1}{r^2}=\frac{k^2}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^4}-\frac{\frac{k^2}{c^2}\left(\frac{dr}{dt}\right)^2}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^4}-\frac{\frac{k^2}{c^2}\left(r\frac{d\phi}{dt}\right)^2}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^4}$$

$$\left(\frac{d^2w}{d\phi^2}\right)^2+\frac{2}{r}\frac{d^2w}{d\phi^2}+\frac{1}{r^2}=\frac{k^2}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^4}-\frac{\frac{k^2}{c^2}\left(\frac{dr}{d\phi}\right)^2}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^2}-\frac{k^2}{m_0^2c^2r^6\left(\frac{d\phi}{dt}\right)^2}$$

$$\left(\frac{d^2w}{d\phi^2}\right)^2+\frac{2}{r}\frac{d^2w}{d\phi^2}+\frac{1}{r^2}=\frac{k^2}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^4}-\frac{\frac{k^2}{c^2}\left(-r^2\frac{dw}{d\phi}\right)^2}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^2}-\frac{k^2}{m_0^2c^2r^6\left(\frac{d\phi}{dt}\right)^2}$$

$$\left(\frac{d^2w}{d\phi^2}\right)^2+\frac{2}{r}\frac{d^2w}{d\phi^2}+\frac{1}{r^2}=\frac{k^2}{m_0^2r^8\left(\frac{d\phi}{dt}\right)^4}-\frac{\frac{k^2}{c^2}\left(\frac{dw}{d\phi}\right)^2}{m_0^2r^4\left(\frac{d\phi}{dt}\right)^2}-\frac{k^2}{m_0^2c^2r^6\left(\frac{d\phi}{dt}\right)^2}$$

In this we will consider constant the Newton's angular moment in the form:

$$L=r^2\frac{d\phi}{dt} \tag{19.53}$$

That it is really the known theoretical angular moment.

$$\left(\frac{d^2w}{d\phi^2}\right)^2+\frac{2}{r}\frac{d^2w}{d\phi^2}+\frac{1}{r^2}=\frac{k^2}{m_0^2L^4}-\frac{k^2}{m_0^2c^2L^2}\left(\frac{dw}{d\phi}\right)^2-\frac{k^2}{m_0^2c^2r^2L^2}$$

$$\left(\frac{d^2w}{d\phi^2}\right)^2+2\frac{d^2w}{d\phi^2}2w+w^2=\frac{k^2}{m_0^2L^4}-\frac{k^2}{m_0^2c^2L^2}\left(\frac{dw}{d\phi}\right)^2-\frac{k^2}{m_0^2c^2L^2}w^2$$

$$\left(\frac{d^2w}{d\phi^2}\right)^2+2\frac{d^2w}{d\phi^2}w+w^2=B-A\left(\frac{dw}{d\phi}\right)^2-Aw^2$$

$$\left(\frac{d^2w}{d\phi^2}\right)^2+2\frac{d^2w}{d\phi^2}w+A\left(\frac{dw}{d\phi}\right)^2+(A+1)w^2-B=\text{zero} \tag{19.54}$$

Where we have:

$$A=\frac{k^2}{m_0^2c^2L^2} \tag{19.55}$$

$$B=\frac{k^2}{m_0^2L^4} \tag{19.56}$$

The equation 19.54 has as solution:

$$w=\frac{1}{\varepsilon D}\left[1-\varepsilon\cos\left(\phi\sqrt{1+A}+\phi_0\right)\right]\Rightarrow w=\frac{1}{\varepsilon D}\left[1-\varepsilon\cos(\phi Q)\right] \qquad 19.57$$

Where we consider $\phi_0=zero$.

It is denominated in 19.57 $Q^2=1+A$. 19.58

The equation 19.58 is function only of A demonstrating the intrinsic union between the variation of the mass with the variation of the energy in the time, because both as already described, participate in the relativistic force 19.06 in this relies the essential difference between the mass and the electric charge that is invariable and indivisible in the electromagnetic theory.

From 19.57 we obtain the ray of a conical:

$$r=\frac{1}{w}=\frac{\varepsilon D}{1-\varepsilon\cos\left(\phi\sqrt{1+A}\right)}\Rightarrow r=\frac{\varepsilon D}{1-\varepsilon\cos(\phi Q)} \qquad 19.59$$

Where ε is the eccentricity and D the directory distance of the focus.

Deriving 19.57 we have $\frac{dw}{d\phi}=\frac{Q\,sen(\phi Q)}{D}$ 19.60

That derived results in $\frac{d^2w}{d\phi^2}=\frac{Q^2\cos(\phi Q)}{D}$ 19.61

Applying in 19.54 the variables we have:

$$\left(\frac{d^2w}{d\phi^2}\right)^2+2\frac{d^2w}{d\phi^2}w+A\left(\frac{dw}{d\phi}\right)^2+(A+1)w^2-B=zero$$

$$\frac{Q^4\cos^2(\phi Q)}{D^2}+2\frac{Q^2\cos(\phi Q)}{D}\left[\frac{1-\varepsilon\cos(\phi Q)}{\varepsilon D}\right]+A\frac{Q^2 sen^2(\phi Q)}{D^2}+(A+1)\left[\frac{1-\varepsilon\cos(\phi Q)}{\varepsilon D}\right]^2-B=zero \qquad 19.62$$

$$\frac{Q^4\cos^2(\phi Q)}{D^2}+2\frac{Q^2\cos(\phi Q)}{\varepsilon D^2}-2\frac{Q^2\cos^2(\phi Q)}{D^2}+A\frac{Q^2}{D^2}-A\frac{Q^2\cos^2(\phi Q)}{D^2}+(A+1)\left[\frac{1-\varepsilon\cos(\phi Q)}{\varepsilon D}\right]^2-B=zero$$

$$\frac{Q^4\cos^2(\phi Q)}{D^2}+2\frac{Q^2\cos(\phi Q)}{\varepsilon D^2}-2\frac{Q^2\cos^2(\phi Q)}{D^2}+A\frac{Q^2}{D^2}-A\frac{Q^2\cos^2(\phi Q)}{D^2}+\frac{(A+1)}{\varepsilon^2D^2}-2\frac{(A+1)\cos(\phi Q)}{\varepsilon D^2}+\frac{(A+1)\cos^2(\phi Q)}{D^2}-B=zero$$

$$\left(Q^4-2Q^2-AQ^2+A+1\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{2Q^2}{\varepsilon D}-\frac{2A}{\varepsilon D}-\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{AQ^2}{D^2}+\frac{(A+1)}{\varepsilon^2D^2}-B=zero \qquad 19.63$$

In this applying in the first parenthesis $Q^2=1+A$ we have:

$$\left(Q^4-2Q^2-AQ^2+A+1\right)=\left[(1+A)^2-2(1+A)-A(1+A)+A+1\right]=\left(1+2A+A^2-2-2A-A-A^2+A+1\right)=zero$$

In 19.63 applying in the second parenthesis $Q^2=1+A$ we have:

$$\left(\frac{2Q^2}{\varepsilon D}-\frac{2A}{\varepsilon D}-\frac{2}{\varepsilon D}\right)=\left[\frac{2(1+A)}{\varepsilon D}-\frac{2A}{\varepsilon D}-\frac{2}{\varepsilon D}\right]=zero$$

The rest of the equation 19.63 is therefore:

$$\frac{AQ^2}{D^2}+\frac{(A+1)}{\varepsilon^2 D^2}-B=zero \qquad 19.64$$

The data of the elliptic orbit of the planet Mercury is [1]:

Eccentricity of the orbit $\varepsilon=0{,}206$.

Larger semi-axis = a = 5,79.10^{10}m.

Smaller semi-axis $b=a\sqrt{1-\varepsilon^2}=5{,}79.10^{10}\sqrt{1-0{,}206^2}=56658160305{,}80m$.

$$\varepsilon D=a(1-\varepsilon^2)=5{,}79.10^{10}(1-0{,}206^2)=55442955600{,}00m.$$

$$D=\frac{a(1-\varepsilon^2)}{\varepsilon}=\frac{5{,}79.10^{10}(1-0{,}206^2)}{0{,}206}=269140561165{,}00m.$$

The orbital period of the Earth (PT) and Mercury (PM) around the Sun in seconds are:

$$PT=3{,}16.10^7 s.$$

$$PM=7{,}60.10^6 s.$$

The number of turns that Mercury (m_o) makes around the Sun (M_o) in one century is, therefore:

$$N=100\frac{3{,}16.10^7}{7{,}60.10^6}=415{,}79. \qquad 19.65$$

Theoretical angular moment of Mercury:

$$L^2=\left(r^2\frac{d\phi}{dt}\right)^2=GM_o a(1-\varepsilon^2)=6{,}67.10^{-11}1{,}98.10^{30}5{,}79.10^{10}(1-0{,}206^2)=7{,}322129374.10^{30} \qquad 19.66$$

$$A=\frac{(GM_o m_o)^2}{m_o^2 c^2 L^2}=\frac{(GM_o)^2}{c^2 L^2}=\frac{(6{,}67.10^{-11})^2(1{,}98.10^{30})^2}{(3{,}0.10^8)^2(7{,}32.10^{30})}=2{,}65.10^{-8}. \qquad 19.67$$

$$B=\frac{(GM_o m_o)^2}{m_o^2 L^4}=\frac{(GM_o)^2}{L^4}=\frac{(6{,}67.10^{-11})^2(1{,}98.10^{30})^2}{(7{,}32.10^{30})^2}=3{,}25.10^{-22} \qquad 19.68$$

$$Q=\sqrt{1+A}=\sqrt{1+2{,}63.10^{-8}}=1{,}00000001323 \qquad 19.69$$

Applying the numeric data with several decimal numbers to the rest of the equation 19.63 we have:

$$\frac{AQ^2}{D^2}+\frac{(A+1)}{\varepsilon^2 D^2}-B=\frac{2{,}65.10^{-8}(1{,}00000001323)^2}{(269140561165{,}00)^2}+\frac{2{,}65.10^{-8}+1}{(55442955600{,}00)^2}-3{,}25.10^{-22}=8{,}976.10^{-30} \qquad 19.70$$

Result that we can consider null.

We will obtain the relativistic angular moment of the rest of the equation 19.63 in this applying the variables we have:

$$\frac{AQ^2}{D^2}+\frac{(A+1)}{\varepsilon^2 D^2}-B=\frac{(GM_0)^2}{c^2L^2D^2}\left[1+\frac{(GM_0)^2}{c^2L^2}\right]+\frac{1}{\varepsilon^2D^2}\left[1+\frac{(GM_0)^2}{c^2L^2}\right]-\frac{(GM_0)^2}{L^4}=zero \qquad 19.71$$

$$\varepsilon^2L^2(GM_0)^2\left[1+\frac{(GM_0)^2}{c^2L^2}\right]+L^4c^2\left[1+\frac{(GM_0)^2}{c^2L^2}\right]-c^2\varepsilon^2D^2(GM_0)^2=zero$$

$$\varepsilon^2L^2(GM_0)^2+\varepsilon^2L^2(GM_0)^2\frac{(GM_0)^2}{c^2L^2}+L^4c^2+L^4c^2\frac{(GM_0)^2}{c^2L^2}-c^2\varepsilon^2D^2(GM_0)^2=zero$$

$$\varepsilon^2L^2(GM_0)^2+\varepsilon^2\frac{(GM_0)^4}{c^2}+L^4c^2+L^2(GM_0)^2-c^2\varepsilon^2D^2(GM_0)^2=zero$$

$$c^2L^4+(1+\varepsilon^2)(GM_0)^2L^2+\varepsilon^2\frac{(GM_0)^4}{c^2}-c^2\varepsilon^2D^2(GM_0)^2=zero \qquad 19.72$$

$$L^2=\frac{-(1+\varepsilon^2)(GM_0)^2\pm\sqrt{\left[(1+\varepsilon^2)(GM_0)^2\right]^2-4c^2\left[\varepsilon^2\frac{(GM_0)^4}{c^2}-c^2\varepsilon^2D^2(GM_0)^2\right]}}{2c^2}$$

$$L^2=\frac{-(1+\varepsilon^2)(GM_0)^2\pm\sqrt{(1+\varepsilon^2)^2(GM_0)^4-4\varepsilon^2(GM_0)^4+4c^4\varepsilon^2D^2(GM_0)^2}}{2c^2}$$

$$L^2=\frac{-(1+\varepsilon^2)(GM_0)^2\pm\sqrt{(1+2\varepsilon^2+\varepsilon^4)(GM_0)^4-4\varepsilon^2(GM_0)^4+4c^4\varepsilon^2D^2(GM_0)^2}}{2c^2}$$

$$L^2=\frac{-(1+\varepsilon^2)(GM_0)^2\pm\sqrt{(GM_0)^4+2\varepsilon^2(GM_0)^4+\varepsilon^4(GM_0)^4-4\varepsilon^2(GM_0)^4+4c^4\varepsilon^2D^2(GM_0)^2}}{2c^2}$$

$$L^2=\frac{-(1+\varepsilon^2)(GM_0)^2\pm\sqrt{(GM_0)^4+\varepsilon^4(GM_0)^4-2\varepsilon^2(GM_0)^4+4c^4\varepsilon^2D^2(GM_0)^2}}{2c^2}$$

$$L^2=\frac{-(1+\varepsilon^2)(GM_0)^2+\sqrt{(1-\varepsilon^2)^2(GM_0)^4+4c^4\varepsilon^2D^2(GM_0)^2}}{2c^2}=7{,}3221292783\cdot10^{30} \qquad 19.73$$

This last equation has the exclusive property of relating the speed c to the denominated relativistic angular moment that is smaller than the theoretical angular moment 19.66.

The variation of the relativistic angular moment in relation to the theoretical angular moment is very small and given by:

$$\Delta L=\frac{7{,}3221292783\cdot10^{30}-7{,}322129374\cdot10^{30}}{7{,}322129374\cdot10^{30}}=-1{,}38\cdot10^{-8}=\frac{-1}{72.503.509{,}00} \qquad 19.74$$

That demonstrates the accuracy of the principle of constancy of the speed of the light.

In reality, the equation 19.06 provides a secular retrocession perihelion of Mercury, which is given by in

$$\Delta\phi = 2\pi 415{,}79\left(\frac{1}{Q}-1\right) = 2\pi 415{,}79\left(-0{,}00000001323\right) = -3{,}46 10^{-5} rad \qquad 19.75$$

Converting for the second we have:

$$\Delta\phi = \frac{-3{,}46 10^{-5}.180{,}00.3.600{,}00}{\pi} = -7{,}13'' . \qquad 19.76$$

This retrocession, is not expected in Newtonian theory is due to relativistic variation of mass and energy and is shrouded in total observed precession of 5599. "

§§19 Advance of Mercury's perihelion of 42.79"

If we write the equation for the gravitational relativity energy E_R covering the terms for the kinetic energy, the potential energy E_p and the resting energy:

$$E_R = m_o c^2 \left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1 \right) + E_p + m_o c^2 = \frac{m_o c^2}{\sqrt{1-\frac{u^2}{c^2}}} + E_p. \qquad 19.77$$

Being the conservative the gravitational force its energy is constant. Assuming then that in 19.77 when the radius tends to infinite, the speed and potential energy tends to zero, resulting then:

$$E_R = \frac{m_o c^2}{\sqrt{1-\frac{u^2}{c^2}}} + E_p = m_o c^2 \qquad 19.78$$

Writing the equation to the Newton's gravitation energy E_N having the correspondent Newton's terms to the 19.77:

$$E_N = \frac{m_o u^2}{2} - \frac{k}{r} + m_o c^2 = m_o c^2 \qquad 19.79$$

Where $\frac{m_o u^2}{2}$ is the kinetic energy, $\frac{-k}{r}$ the potential energy and $m_o c^2$ the resting energy or better saying the inertial energy.

From this 19.79 we have:

$$\frac{m_o u^2}{2} - \frac{k}{r} + m_o c^2 = m_o c^2 \Rightarrow \frac{m_o u^2}{2} = \frac{k}{r} \Rightarrow u^2 = \frac{2k}{m_o r} = \frac{2GM_o m_o}{m_o r} \Rightarrow u^2 = \frac{2GM_o}{r} \qquad 19.80$$

Deriving 19.79 we have:

$$\frac{dE_N}{dt} = \frac{d}{dt}\left(\frac{m_o u^2}{2} - \frac{k}{r} + m_o c^2 \right) = zer($$

$$\frac{m_o 2u}{2}\frac{du}{dt} + \frac{k}{r^2}\frac{dr}{dt} = zer($$

$$u\frac{du}{dt} = \frac{-k}{m_o r^2}\frac{dr}{dt} = \frac{-GM_o}{r^2}\frac{dr}{dt}$$

$$u\frac{du}{dt} = \frac{-GM_o}{r^2}\frac{dr}{dt}$$

$$u\frac{du}{dr} = \frac{-GM_o}{r^2} \qquad 19.81$$

Making the relativity energy 19.78 equal to the Newton's energy 19.79 we have:

$$E_R = E_N \Rightarrow \frac{m_o c^2}{\sqrt{1-\frac{u^2}{c^2}}} + E_p = \frac{m_o u^2}{2} - \frac{k}{r} + m_o c^2 \qquad 19.82$$

$$\frac{m_o c^2}{m_o\sqrt{1-\frac{u^2}{c^2}}} + \frac{E_p}{m_o} = \frac{m_o u^2}{m_o 2} - \frac{GM_o m_o}{m_o r} + \frac{m_o c^2}{m_o} \qquad 19.83$$

In that denominating the relativity potential (φ) as:

$$\varphi = \frac{E_p}{m_o} \qquad 19.84$$

We have:

$$\frac{c^2}{\sqrt{1-\frac{u^2}{c^2}}} + \varphi = \frac{u^2}{2} - \frac{GM_o}{r} + c^2$$

$$\varphi = \frac{u^2}{2} - \frac{GM_o}{r} + c^2 - \frac{c^2}{\sqrt{1-\frac{u^2}{c^2}}} \qquad 19.85$$

In this one replacing the approximation:

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \approx 1 + \frac{u^2}{2c^2} \qquad 19.86$$

We have:

$$\varphi = \frac{u^2}{2} - \frac{GM_o}{r} + c^2 - c^2\left(1 + \frac{u^2}{2c^2}\right)$$

That simplified results in the Newton's potential:

$$\varphi = \frac{u^2}{2} - \frac{GM_o}{r} + c^2 - c^2 - \frac{u^2}{2} = \frac{-GM_o}{r} \qquad 19.87$$

Replacing 19.84 and the relativity potential 19.85 in the relativity energy 19.78:

$$E_R = \frac{m_o c^2}{\sqrt{1-\frac{u^2}{c^2}}} + m_o\left(\frac{u^2}{2} - \frac{GM_o}{r} + c^2 - \frac{c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right) \qquad 19.88$$

We have the Newton's energy 19.79:

$$E_N = \frac{m_o u^2}{2} - \frac{GM_o m_o}{r} + m_o c^2$$

Deriving the relativity potential 19.85 we have the relativity gravitational acceleration modulus exactly as in the Newton's theory:

$$a=\frac{-d\varphi}{dr}$$

$$a=\frac{-d\varphi}{dr}=\frac{-d}{dr}\left(\frac{u^2}{2}-\frac{GM_o}{r}+c^2-\frac{c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$a=\frac{-d}{dr}\left(\frac{u^2}{2}-\frac{GM_o}{r}+c^2\right)-\frac{d}{dr}\left(-\frac{c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

Where we have:

$\frac{-d}{dr}\left(\frac{u^2}{2}-\frac{GM_o}{r}+c^2\right)=\frac{-d}{dr}\left(\frac{E_N}{m_o}\right)=zero$. Because the term to be derived is the Newton's energy divided by m_o that is $\frac{E_N}{m_o}=\frac{u^2}{2}-\frac{GM_o}{r}+c^2$ that is constant, resulting then in:

$$a=-\frac{d}{dr}\left(-\frac{c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$a=-\left[-\frac{u}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\frac{du}{dr}\right]$$

In this one applying 19.81 we have:

$$a=\frac{-1}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\frac{GM_o}{r^2} \qquad 19.89$$

The vector acceleration is given by 19.05:

$$\vec{a}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]\hat{r}+\left[2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right]\hat{\phi}$$

The relativity gravitational acceleration modulus 19.89 is equal to the component of the vector radius ($\hat{r}$) thus we have:

$$a=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]=\frac{-1}{\left(1-\frac{u^2}{c^2}\right)^{\frac{3}{2}}}\frac{GM_\circ}{r^2} \quad 19.90$$

Being null the transversal acceleration we have:

$$\left[2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right]\hat{\phi}=zero \quad 19.91$$

$$2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}=zero$$

That is equal to the derivative of the constant angular momentum $L=r^2\frac{d\phi}{dt}$ 19.92

$$\frac{dL}{dt}=\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\right)=2r\frac{dr}{dt}\frac{d\phi}{dt}+r^2\frac{d^2\phi}{dt^2}=zero \quad 19.93$$

Rewriting some equations already described we have:

$$w=\frac{1}{r}$$

$$dw=\frac{\partial w}{\partial r}dr\Rightarrow dw=\frac{-1}{r^2}dr$$

$\frac{dw}{d\phi}=\frac{-1}{r^2}\frac{dr}{d\phi}$ or $\frac{dr}{d\phi}=-r^2\frac{dw}{d\phi}$ and $\frac{dw}{dt}=\frac{-1}{r^2}\frac{dr}{dt}$

$$\frac{dr}{dt}=\frac{d\phi}{dt}\frac{dt}{d\phi}\frac{dr}{dt}=\frac{L}{r^2}\frac{dr}{d\phi}=\frac{-L}{r^2}r^2\frac{dw}{d\phi}\Rightarrow\frac{dr}{dt}=-L\frac{dw}{d\phi}$$

$$\frac{d^2r}{dt^2}=\frac{d}{dt}\left(\frac{dr}{dt}\right)=\frac{d\phi}{dt}\frac{dt}{d\phi}\frac{d}{dt}\left(-L\frac{dw}{d\phi}\right)=\frac{L}{r^2}\frac{d}{d\phi}\left(-L\frac{dw}{d\phi}\right)=\frac{-L^2}{r^2}\frac{d^2w}{d\phi^2} \quad 19.94$$

From 19.90 we have:

$$\left(1-\frac{3u^2}{2c^2}\right)\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]=\frac{-GM_\circ}{r^2}$$

In this one we 19.94 the speed of 19.80 and the angular momentum we have:

$$\left[1-\frac{3}{2c^2}\left(\frac{2GM_\circ}{r}\right)\right]\left[\frac{-L^2}{r^2}\frac{d^2w}{d\phi^2}-r\left(\frac{L}{r^2}\right)^2\right]=-\frac{GM_\circ}{r^2}$$

$$\left(1-\frac{3GM_\circ}{c^2}\frac{1}{r}\right)\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{GM_\circ}{L^2}$$

$$\left(1-\frac{3GM_{\circ}}{c^2}\frac{1}{r}\right)\frac{d^2w}{d\phi^2}+\left(1-\frac{3Gm_{\circ}}{c^2}\frac{1}{r}\right)\frac{1}{r}=\frac{GM_{\circ}}{L^2}$$

$$\frac{d^2w}{d\phi^2}-\frac{3GM_{\circ}}{c^2}\frac{d^2w}{d\phi^2}\frac{1}{r}+\frac{1}{r}-\frac{3GM_{\circ}}{c^2}\frac{1}{r^2}-\frac{GM_{\circ}}{L^2}=zero$$

$$\frac{d^2w}{d\phi^2}-A\frac{d^2w}{d\phi^2}\frac{1}{r}+\frac{1}{r}-A\frac{1}{r^2}-B=zero$$

$$\frac{d^2w}{d\phi^2}-A\frac{d^2w}{d\phi^2}w+w-Aw^2-B=zero$$

$$\frac{d^2w}{d\phi^2}-A\frac{d^2w}{d\phi^2}w-Aw^2+w-B=zero \qquad 19.95$$

Where we have:

$$A=\frac{3GM_{\circ}}{c^2} \quad B=\frac{GM_{\circ}}{L^2} \qquad 19.96$$

The solution to the differential equation 19.95 is:

$$w=\frac{1}{\varepsilon D}\left[1-\varepsilon\cos(\phi Q+\phi_{\circ})\right]\Rightarrow w=\frac{1}{\varepsilon D}\left[1-\varepsilon\cos(\phi Q)\right]. \qquad 19.97$$

Where we consider $\phi_{\circ}=zero$

Then the radius is given by:

$$r=\frac{1}{w}=\frac{\varepsilon D}{1-\varepsilon\cos(\phi Q)}\Rightarrow r=\frac{\varepsilon D}{1-\varepsilon\cos(\phi Q)} \qquad 19.98$$

Where ε is the eccentricity and D the focus distance to the directory.

Deriving 19.97 we have $\frac{dw}{d\phi}=\frac{Q\,sen(\phi Q)}{D}$ and $\frac{d^2w}{d\phi^2}=\frac{Q^2\cos(\phi Q)}{D}$ 19.99

Applying the derivatives in 19.95 we have:

$$\frac{d^2w}{d\phi^2}-A\frac{d^2w}{d\phi^2}w-Aw^2+w-B=zero$$

$$\frac{Q^2\cos(\phi Q)}{D}-\frac{AQ^2\cos(\phi Q)}{D}\frac{1}{\varepsilon D}\left[1-\varepsilon\cos(\phi Q)\right]-\frac{A}{\varepsilon^2D^2}\left[1-\varepsilon\cos(\phi Q)\right]^2+\frac{1}{\varepsilon D}\left[1-\varepsilon\cos(\phi Q)\right]-B=zero$$

$$\frac{Q^2\cos(\phi Q)}{D}-\frac{AQ^2\cos(\phi Q)}{\varepsilon D^2}\left[1-\varepsilon\cos(\phi Q)\right]-\frac{A}{\varepsilon^2D^2}\left[1-2\varepsilon\cos(\phi Q)+\varepsilon^2\cos^2(\phi Q)\right]+\left[\frac{1}{\varepsilon D}-\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q)\right]-B=zero$$

$$\frac{Q^2\cos(\phi Q)}{D}-\frac{AQ^2\cos(\phi Q)}{\varepsilon D^2}+\frac{AQ^2\cos(\phi Q)}{\varepsilon D^2}\varepsilon\cos(\phi Q)-$$

$$-\frac{A}{\varepsilon^2D^2}+\frac{A}{\varepsilon^2D^2}2\varepsilon\cos(\phi Q)-\frac{A}{\varepsilon^2D^2}\varepsilon^2\cos^2(\phi Q)+\frac{1}{\varepsilon D}-\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q)-B=zero$$

$$\frac{\cos(\phi Q)}{D}\left(Q^2-\frac{AQ^2}{\varepsilon D}+\frac{2A}{\varepsilon D}-1\right)+\frac{AQ^2\cos^2(\phi Q)}{D^2}-\frac{A\cos^2(\phi Q)}{D^2}-\frac{A}{\varepsilon^2 D^2}+\frac{1}{\varepsilon D}-B=\text{zero}$$

$$\frac{\cos(\phi Q)}{AD}\left(Q^2-\frac{AQ^2}{\varepsilon D}+\frac{2A}{\varepsilon D}-1\right)+\frac{AQ^2\cos^2(\phi Q)}{AD^2}-\frac{A\cos^2(\phi Q)}{AD^2}-\frac{A}{A\varepsilon^2 D^2}+\frac{1}{A\varepsilon D}-\frac{B}{A}=\text{zero}$$

$$\frac{\cos(\phi Q)}{D}\left(\frac{Q^2}{A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}-\frac{1}{A}\right)+\frac{Q^2\cos^2(\phi Q)}{D^2}-\frac{\cos^2(\phi Q)}{D^2}-\frac{1}{\varepsilon^2 D^2}+\frac{1}{A\varepsilon D}-\frac{B}{A}=\text{zero}$$

$$\frac{\cos^2(\phi Q)}{D^2}\left(Q^2-1\right)+\frac{\cos(\phi Q)}{D}\left(\frac{Q^2}{A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}-\frac{1}{A}\right)-\frac{1}{\varepsilon^2 D^2}+\frac{1}{A\varepsilon D}-\frac{B}{A}=\text{zero} \qquad 19.100$$

The coefficient of the squared co-cosine can be considered null because $Q\approx 1$ and D^2 is a very large number:

$$\frac{\cos^2(\phi Q)}{D^2}\left(Q^2-1\right)=\text{zero} \qquad 19.101$$

Resulting from the equation 19.100:

$$\frac{\cos(\phi Q)}{D}\left(\frac{Q^2}{A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}-\frac{1}{A}\right)-\frac{1}{\varepsilon^2 D^2}+\frac{1}{A\varepsilon D}-\frac{B}{A}=\text{zero} \qquad 19.102$$

Due to the unicity of the equation 19.102 we must have the only solution that makes it null simultaneously the parenthesis and the rest of the equation, that is, we must have a unique solution for both the following equations:

$$\frac{Q^2}{A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}-\frac{1}{A}=\text{zero} \quad \text{and} \quad -\frac{1}{\varepsilon^2 D^2}+\frac{1}{A\varepsilon D}-\frac{B}{A}=\text{zero} \qquad 19.103$$

These equations can be written as:

$$[a=b]\Rightarrow\frac{1}{A}-\frac{1}{\varepsilon D}=\frac{1}{Q^2}\left(\frac{1}{A}-\frac{2}{\varepsilon D}\right) \qquad 19.104$$

$$[a=c]\Rightarrow\frac{1}{A}-\frac{1}{\varepsilon D}=\frac{\varepsilon DB}{A} \qquad 19.105$$

In these ones the common term $a=\frac{1}{A}-\frac{1}{\varepsilon D}$ must have a single solution then we have:

$$[b=c]\Rightarrow\frac{1}{Q^2}\left(\frac{1}{A}-\frac{2}{\varepsilon D}\right)=\frac{\varepsilon DB}{A} \qquad 19.106$$

With 19.96 and the theoretical momentum we have:

$$A=\frac{3GM_{\odot}}{c^2} \quad B=\frac{GM_{\odot}}{L^2} \quad L^2=\varepsilon DGM_{\odot} \qquad \varepsilon DB=\frac{\varepsilon DGM_{\odot}}{L^2}=1 \qquad 19.107$$

It is applied in 19.105 and 19.106 resulting in:

$$[a=c] \Rightarrow \frac{1}{A} - \frac{1}{\varepsilon D} = \frac{1}{A} \qquad 19.108$$

$$[b=c] \Rightarrow \frac{1}{Q^2}\left(\frac{1}{A} - \frac{2}{\varepsilon D}\right) = \frac{1}{A} \qquad 19.109$$

From 19.108 we have the mistake made in 19.105:

$$\frac{1}{A} - \frac{1}{\varepsilon D} = \frac{1}{A} \Rightarrow -\frac{1}{\varepsilon D} \approx \text{zero} \qquad 19.110$$

$$-\frac{1}{\varepsilon D} = \frac{-1}{55.442.955.600,00} = -1,80.10^{-11} \approx \text{zero} \qquad 19.111$$

From 19.109 we have Q:

$$\frac{1}{Q^2}\left(\frac{1}{A} - \frac{2}{\varepsilon D}\right) = \frac{1}{A} \Rightarrow Q^2 = 1 - \frac{2A}{\varepsilon D} \Rightarrow Q^2 = 1 - \frac{2}{\varepsilon D}\frac{3GM_{\circ}}{c^2} \qquad 19.112$$

It is applied in 19.104 resulting in 19.110:

$$\frac{1}{A} - \frac{1}{\varepsilon D} = \frac{1}{Q^2}\left(\frac{1}{A} - \frac{2}{\varepsilon D}\right) \Rightarrow \frac{1}{A} - \frac{1}{\varepsilon D} = \frac{1}{\left(1 - \frac{2A}{\varepsilon D}\right)}\left(\frac{1}{A} - \frac{2}{\varepsilon D}\right) \Rightarrow \frac{1}{A} - \frac{1}{\varepsilon D} = \frac{1}{A} \Rightarrow -\frac{1}{\varepsilon D} \approx \text{zero}$$

From 19.112 we have:

$$Q = \sqrt{1 - \frac{6GM_{\circ}}{\varepsilon D c^2}} = \sqrt{1 - \frac{6(6,67.10^{-11})(1,98.10^{30})}{(55.442.955.600,00)(3.10^8)^2}} = 0,999.999.920.59 \qquad 19.113$$

That corresponds to the advance of Mercury’s perihelion in one century of:

$$\sum \Delta\phi = \Delta\phi.415,79 = \left(\frac{1}{Q} - 1\right).1.296.000,00.415,79 = 42,79” \qquad 19.114$$

Calculated in this way:

In one trigonometric turn we have $360 \times 60 \times 60 = 1.296.000,00"$ seconds.

The angle ϕ in seconds ran by the planet in one trigonometric turn is given by:

$$\phi Q = 1.296.000,00 \Rightarrow \phi = \frac{1.296.000,00}{Q}.$$

If $Q > 1,00$ we have a regression. $\phi < 1.296.000,00$.

If $Q < 1,00$ we have an advance. $\phi > 1.296.000,00$.

The angular variation in seconds in one turn is given by:

$$\Delta\phi = \frac{1.296.000,00}{Q} - 1.296.000,00 = \left(\frac{1}{Q} - 1\right) 1.296.000,00$$

If $\Delta\phi < zero$ we have a regression.

If $\Delta\phi > zero$ we have an advance.

In one century we have 415,79 turns that supply a total angular variation of:

$$\sum \Delta\phi = \Delta\phi . 415,79 = \left(\frac{1}{Q} - 1\right) . 1.296.000,00 . 415,79 = 42,79''$$

If $\sum \Delta\phi < zero$ we have a regression.

If $\sum \Delta\phi > zero$ we have an advance.

§20 Inertia

Imagine in an infinite universe totally empty, a point O' which is the beginning of the coordinates of the observer O'. In the cases of the observer O’ being at rest or in uniform motion the law of inertia requires that the spherical electromagnetic waves with speed c issued by a source located at point O' is always observed by O', regardless of time, with spherical speed c and therefore the uniform motion and rest are indistinguishable from each other remain valid in both cases the law of inertia. To the observer O’ the equations of electromagnetic theory describe the spread just like a spherical wave. The image of an object located in O’ will always be centered on the object itself and a beam of light emitted from O' will always remain straight and perpendicular to the spherical waves.

Imagine another point O what will be the beginning of the coordinates of the observer which has the same properties as described for the inertial observer O'.

Obviously two imaginary points without any form of interaction between them remain individually and together perfectly meeting the law of inertia even though there is a uniform motion between them only detectable due to the presence of two observers who will be considered individually in rest, setting in motion the other referential.

The intrinsic properties of these two observers are described by the equations of relativistic transformations.

Note: the infinite universe is one in which any point can be considered the central point of this universe.

(§ 20 electronic translation)

§20 Inertia (clarifications)

Imagine in a totally empty infinite universe a single point O. Due to the uniqueness properties of O a radius of light emitted from O must propagate with velocity c. If this ray propagates in a straight line, then O is defined as the origin of an inertial frame because it is either at rest or in a uniform rectilinear motion. However, in the hypothesis of propagation of the light ray being a curve the movement of O must be interpreted as the origin of an accelerated frame. Therefore the propagation of a ray of light is sufficient to demonstrate whether O is the origin of an inertial frame or accelerated frame.

Now imagine if in the universe described above for the inertial reference frame O there is another inertial frame O' that does not have any kind of physical interaction with O. In the absence of any interaction between O and O' the uniqueness properties are inviolable for both points and rays of light emitted from O and O' have the same velocity c. It is impossible for the velocity of light emitted from O to be different from the velocity of light emitted from O' because each reference exists as if the other did not exist. Being O and O' the origin of inertial frames the propagation of light rays occurs in a straight line with velocity c and the relations between times t and t' of each frame are given by table I.

§21 Advance of Mercury's perihelion of 42.79" calculated with the Undulating Relativity

Assuming $ux = v$

(2.3) $$u'x' = \frac{ux - v}{\sqrt{1 + \frac{v^2}{c^2} - \frac{2vux}{c^2}}} = \frac{v - v}{\sqrt{1 + \frac{v^2}{c^2} - \frac{2vv}{c^2}}} \Rightarrow u'x' = zero$$

$$ux = v \qquad u'x' = zero \qquad 21.01$$

(1.17) $$dt' = dt\sqrt{1 + \frac{v^2}{c^2} - \frac{2vux}{c^2}} = dt\sqrt{1 + \frac{v^2}{c^2} - \frac{2vv}{c^2}} \Rightarrow dt' = dt\sqrt{1 - \frac{v^2}{c^2}}$$

(1.22) $$dt = dt'\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'}{c^2}} = dt'\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'(0)}{c^2}} \Rightarrow dt = dt'\sqrt{1 + \frac{v'^2}{c^2}}$$

$$dt' = dt\sqrt{1 - \frac{v^2}{c^2}} \qquad dt = dt'\sqrt{1 + \frac{v'^2}{c^2}} \qquad 21.02$$

$$\sqrt{1 - \frac{v^2}{c^2}}\sqrt{1 + \frac{v'^2}{c^2}} = 1 \qquad 21.03$$

$$v = \frac{v'}{\sqrt{1 + \frac{v'^2}{c^2}}} \qquad v' = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad 21.04$$

$$dt > dt' \qquad v < v' \qquad vdt = v'dt' \qquad 21.05$$

(1.33) $$\vec{v} = \frac{-\vec{v}'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'u'x'}{c^2}}} = \frac{-\vec{v}'}{\sqrt{1 + \frac{v'^2}{c^2} + \frac{2v'(0)}{c^2}}} \Rightarrow \vec{v} = \frac{-\vec{v}'}{\sqrt{1 + \frac{v'^2}{c^2}}}$$

(1.34) $$\vec{v}' = \frac{-\vec{v}}{\sqrt{1 + \frac{v^2}{c^2} - \frac{2vux}{c^2}}} = \frac{-\vec{v}}{\sqrt{1 + \frac{v^2}{c^2} - \frac{2vv}{c^2}}} \Rightarrow \vec{v}' = \frac{-\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\vec{v} = \frac{-\vec{v}'}{\sqrt{1 + \frac{v'^2}{c^2}}} \qquad -\vec{v}' = \frac{\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad 21.06$$

$$\vec{r} = r\hat{r} = -\vec{r}' \qquad \vec{r}' = -r\hat{r} = -\vec{r} \qquad |\vec{r}| = |\vec{r}'| = r \qquad 21.07$$

$$d\vec{r} = dr\hat{r} + rd\hat{r} = -d\vec{r}' \qquad d\vec{r}' = -dr\hat{r} - rd\hat{r} = -d\vec{r} \qquad 21.08$$

$$\hat{r}d\vec{r} = dr\hat{r}\hat{r} + r\hat{r}d\hat{r} = dr \qquad \hat{r}d\vec{r}' = -dr\hat{r}\hat{r} - r\hat{r}d\hat{r} = -dr \qquad 21.09$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r\hat{r})}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\phi}{dt}\hat{\phi} \qquad v^2 = \vec{v}\vec{v} = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\phi}{dt}\right)^2 \qquad 21.10$$

$$\vec{v}' = \frac{d\vec{r}'}{dt'} = \frac{d(-r\hat{r})}{dt'} = -\left(\frac{dr}{dt'}\hat{r} + r\frac{d\phi}{dt'}\hat{\phi}\right) \qquad v'^2 = \vec{v}'\vec{v}' = \left(\frac{dr}{dt'}\right)^2 + \left(r\frac{d\phi}{dt'}\right)^2 \qquad 21.11$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d^2\vec{r}}{dt^2}=\frac{d^2(r\hat{r})}{dt^2}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]\hat{r}+\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\hat{\phi} \tag{21.12}$$

$$\vec{a}'=\frac{d\vec{v}'}{dt'}=\frac{d^2\vec{r}'}{dt'^2}=\frac{d^2(-r\hat{r})}{dt'^2}=-\left[\frac{d^2r}{dt'^2}-r\left(\frac{d\phi}{dt'}\right)^2\right]\hat{r}-\left(2\frac{dr}{dt'}\frac{d\phi}{dt'}+r\frac{d^2\phi}{dt'^2}\right)\hat{\phi} \tag{21.13}$$

$$-\vec{v}'=\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{21.06}$$

$$-\vec{a}'=\frac{d(-\vec{v}')}{dt'}=\frac{d}{dt'}\left(\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\frac{dt}{dt'}\frac{d}{dt}\left(\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\sqrt{1+\frac{v'^2}{c^2}}\frac{d}{dt}\left(\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{21.14}$$

$$-\vec{a}'=-\frac{d\vec{v}'}{dt'}=\sqrt{1+\frac{v'^2}{c^2}}\frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left[\sqrt{1-\frac{v^2}{c^2}}\frac{d\vec{v}}{dt}-\vec{v}\frac{d}{dt}\left(\sqrt{1-\frac{v^2}{c^2}}\right)\right]$$

$$-\vec{a}'=-\frac{d\vec{v}'}{dt'}=\sqrt{1+\frac{v'^2}{c^2}}\frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left[\sqrt{1-\frac{v^2}{c^2}}\frac{d\vec{v}}{dt}-\vec{v}\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}-\frac{2}{2}=\frac{-1}{2}}\left(\frac{-2v}{c^2}\frac{dv}{dt}\right)\right]$$

$$-\vec{a}'=-\frac{d\vec{v}'}{dt'}=\sqrt{1+\frac{v'^2}{c^2}}\frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left(\sqrt{1-\frac{v^2}{c^2}}\frac{d\vec{v}}{dt}+\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}v\frac{dv}{dt}\frac{\vec{v}}{c^2}\right)$$

$$-\vec{a}'=-\frac{d\vec{v}'}{dt'}=\sqrt{1+\frac{v'^2}{c^2}}\frac{1}{\left(1-\frac{v^2}{c^2}\right)}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}\frac{d\vec{v}}{dt}+\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}v\frac{dv}{dt}\frac{\vec{v}}{c^2}\right)$$

$$-\vec{a}'=-\frac{d\vec{v}'}{dt'}=\sqrt{1+\frac{v'^2}{c^2}}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\frac{d\vec{v}}{dt}+v\frac{dv}{dt}\frac{\vec{v}}{c^2}\right]$$

$$-m'\vec{a}'=\frac{-m_o\vec{a}'}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{-m_o}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{d\vec{v}'}{dt'}=\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\frac{d\vec{v}}{dt}+v\frac{dv}{dt}\frac{\vec{v}}{c^2}\right]$$

$$F'=-m'\vec{a}'=\frac{-m_o\vec{a}'}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{-m_o}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{d\vec{v}'}{dt'} \tag{21.15}$$

$$\vec{F}=\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\frac{d\vec{v}}{dt}+v\frac{dv\,\vec{v}}{dt\,c^2}\right] \qquad (=19.06) \tag{21.16}$$

$$\vec{F}'=-m\vec{a}'=\frac{-m_o\vec{a}'}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{-m_o}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{d\vec{v}'}{dt'}=\vec{F}=\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\frac{d\vec{v}}{dt}+v\frac{dv\,\vec{v}}{dt\,c^2}\right] \tag{21.17}$$

$$E_k=\int\vec{F}'\left(-d\vec{r}'\right)=\int\vec{F}.d\vec{r}=\int\frac{-k}{r^2}\hat{r}d\vec{r} \tag{21.18}$$

$$E_k=\int\vec{F}'\left(-d\vec{r}'\right)=\int\vec{F}.d\vec{r}=\int\frac{-m_o}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{d\vec{v}'}{dt'}\left(-d\vec{r}'\right)=\int\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\frac{d\vec{v}}{dt}+v\frac{dv\,\vec{v}}{dt\,c^2}\right]d\vec{r}=\int\frac{-k}{r^2}\hat{r}.d\vec{r} \tag{21.19}$$

$$E_k=\int\frac{m_o}{\sqrt{1+\frac{v'^2}{c^2}}}d\vec{v}'\frac{d\vec{r}'}{dt'}=\int\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)d\vec{v}\frac{d\vec{r}}{dt}+v\,dv\frac{d\vec{r}}{dt\,c^2}\vec{v}\right]=\int\frac{-k}{r^2}\hat{r}.d\vec{r}$$

$$E_k=\int\frac{m_o d\vec{v}'\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}=\int\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)d\vec{v}\vec{v}+v\,dv\frac{\vec{v}\vec{v}}{c^2}\right]=\int\frac{-k}{r^2}\hat{r}.d\vec{r}$$

$$E_k=\int\frac{m_o v' dv'}{\sqrt{1+\frac{v'^2}{c^2}}}=\int\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)v\,dv+v\,dv\frac{v^2}{c^2}\right]=\int\frac{-k}{r^2}dr$$

$$E_k=\int\frac{m_o v' dv'}{\sqrt{1+\frac{v'^2}{c^2}}}=\int\frac{m_o v\,dv}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left(1-\frac{v^2}{c^2}+\frac{v^2}{c^2}\right)=\int\frac{-k}{r^2}dr$$

$$E_k=\int\frac{m_o v' dv'}{\sqrt{1+\frac{v'^2}{c^2}}}=\int\frac{m_o v\,dv}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}=\int\frac{-k}{r^2}dr \qquad dE_k=\frac{m_o v' dv'}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{m_o v\,dv}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}=\frac{-k}{r^2}dr \tag{21.20}$$

$$E_k=m_o c^2\sqrt{1+\frac{v'^2}{c^2}}=\frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{k}{r}+constant \tag{21.21}$$

$$E_R=m_o c^2\sqrt{1+\frac{v'^2}{c^2}}-\frac{k}{r}=constant \qquad E_R=\frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}-\frac{k}{r}=constant \tag{21.22}$$

$$E_R=\frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}-\frac{k}{r}=m_o c^2+\frac{m_o v^2}{2}-\frac{k}{r} \qquad E_R=\frac{m_o c^2}{\sqrt{1-\frac{(0)^2}{c^2}}}-\frac{k}{\infty}=m_o c^2 \tag{21.23}$$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{E_R}{m_o c^2}+\frac{k}{m_o c^2}\frac{1}{r} \qquad H=\frac{E_R}{m_o c^2} \qquad A=\frac{k}{m_o c^2}=\frac{GMm_o}{m_o c^2}=\frac{GM_o}{c^2} \qquad 21.24$$

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=H+A\frac{1}{r} \qquad \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}=\left(H+A\frac{1}{r}\right)^3 \qquad 21.25$$

$$\vec{L}=\vec{r}\times\vec{v}=r\hat{r}\times\left(\frac{dr}{dt}\hat{r}+r\frac{d\phi}{dt}\hat{\phi}\right)=r^2\frac{d\phi}{dt}\left(\hat{r}\times\hat{\phi}\right)=r^2\frac{d\phi}{dt}\hat{k} \qquad 21.26$$

$$\vec{L}=\vec{r}\times\vec{v}=\vec{r}\times\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}=r\hat{r}\times\frac{-1}{\sqrt{1+\frac{v'^2}{c^2}}}\left[-\left(\frac{dr}{dt'}\hat{r}+r\frac{d\phi}{dt'}\hat{\phi}\right)\right]=\frac{1}{\sqrt{1+\frac{v'^2}{c^2}}}r^2\frac{d\phi}{dt'}\left(\hat{r}\times\hat{\phi}\right)=r^2\frac{d\phi}{dt}\hat{k} \qquad 21.26$$

$$\vec{L}=r^2\frac{d\phi}{dt}\hat{k}=L\hat{k}=\text{ constant} \qquad L=r^2\frac{d\phi}{dt} \qquad 21.27$$

$$dE_k=\frac{m_o v' dv'}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{m_o v dv}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}=\frac{-k}{r^2}dr=\frac{-k}{r^2}\hat{r}.d\vec{r} \qquad 21.20$$

$$\frac{dE_k}{dt}=\vec{F}\vec{v}=\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\vec{v}\frac{d\vec{v}}{dt}=\frac{-k}{r^2}\hat{r}.\frac{d\vec{r}}{dt}=\frac{-k}{r^2}\hat{r}.\vec{v}$$

$$\vec{F}=\frac{m_o\vec{a}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}=\frac{-k}{r^2}\hat{r} \qquad 21.28$$

$$\vec{F}=\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left\{\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]\hat{r}+\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\hat{\phi}\right\}=\frac{-k}{r^2}\hat{r} \qquad 21.29$$

$$\vec{F}_{\hat{\phi}}=\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\hat{\phi}=zer(\qquad 21.30$$

$$\vec{F}_{\hat{r}}=\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]\hat{r}=\frac{-k}{r^2}\hat{r} \qquad 21.31$$

$$\frac{d\phi}{dt}=\frac{L}{r^2} \qquad \frac{dr}{dt}=-L\frac{dw}{d\phi} \qquad \frac{d^2r}{dt^2}=\frac{-L^2}{r^2}\frac{d^2w}{d\phi^2} \qquad \frac{d^2\phi}{dt^2}=\frac{2L^2}{r^3}\frac{dw}{d\phi} \qquad 21.32$$

$$F_{\hat{r}}=\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{-L^2}{r^2}\frac{d^2w}{d\phi^2}-r\left(\frac{L}{r^2}\right)^2\right]\hat{r}=\frac{-k}{r^2}\hat{r} \qquad 21.33$$

$$\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left(\frac{-L^2}{r^2}\frac{d^2w}{d\phi^2}-\frac{L^2}{r^3}\right)=\frac{-GM_o}{r^2}$$

$$\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)\left(\frac{-L^2}{r^2}\right)=\frac{-GM_o}{r^2}$$

$$\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{GM_o}{L^2} \qquad 21.34$$

$$\left(H+A\frac{1}{r}\right)^3\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{GM_o}{L^2} \qquad 21.35$$

$$\left(H+3A\frac{1}{r}\right)\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{GM_o}{L^2}$$

$$H\frac{d^2w}{d\phi^2}+H\frac{1}{r}+3A\frac{d^2w}{d\phi^2}\frac{1}{r}+3A\frac{1}{r^2}=\frac{GM_o}{L^2}$$

$$H\frac{d^2w}{d\phi^2}+Hw+3A\frac{d^2w}{d\phi^2}w+3Aw^2-\frac{GM_o}{L^2}=zero$$

$$H=\frac{E_R}{m_oc^2} \qquad A=\frac{k}{m_oc^2}=\frac{GM_om_o}{m_oc^2}=\frac{GM_o}{c^2} \qquad B=\frac{GM_o}{L^2} \qquad 21.36$$

$$H\frac{d^2w}{d\phi^2}+Hw+3A\frac{d^2w}{d\phi^2}w+3Aw^2-B=zero \qquad 21.37$$

$$w=\frac{1}{r}=\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right] \qquad \frac{dw}{d\phi}=\frac{-Q\,sen(\phi Q)}{D} \qquad \frac{d^2w}{d\phi^2}=\frac{-Q^2\cos(\phi Q)}{D} \qquad 21.38$$

$$H\frac{-Q^2\cos(\phi Q)}{D}+H\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right]+3A\frac{-Q^2\cos(\phi Q)}{D}\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right]+3A\left\{\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right]\right\}^2-B=zero \qquad 21.39$$

$$-Q^2H\frac{\cos(\phi Q)}{D}+H\frac{1}{\varepsilon D}+H\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q)-\frac{3Q^2A}{\varepsilon D}\frac{\cos(\phi Q)}{D}\left[1+\varepsilon\cos(\phi Q)\right]+\frac{3A}{\varepsilon^2D^2}\left[1+2\varepsilon\cos(\phi Q)+\varepsilon^2\cos^2(\phi Q)\right]-B=zero$$

$$-Q^2H\frac{\cos(\phi Q)}{D}+H\frac{1}{\varepsilon D}+H\frac{\cos(\phi Q)}{D}-\frac{3Q^2A}{\varepsilon D}\frac{\cos(\phi Q)}{D}-\frac{3Q^2A}{\varepsilon D}\frac{\cos(\phi Q)}{D}\varepsilon\cos(\phi Q)+$$
$$+\frac{3A}{\varepsilon^2D^2}+\frac{3A}{\varepsilon^2D^2}2\varepsilon\cos(\phi Q)+\frac{3A}{\varepsilon^2D^2}\varepsilon^2\cos^2(\phi Q)-B=zero$$

$$-Q^2H\frac{\cos(\phi Q)}{D}+H\frac{1}{\varepsilon D}+H\frac{\cos(\phi Q)}{D}-\frac{3Q^2A}{\varepsilon D}\frac{\cos(\phi Q)}{D}-3Q^2A\frac{\cos^2(\phi Q)}{D^2}+$$

$$+\frac{3A}{\varepsilon^2D^2}+\frac{6A}{\varepsilon D}\frac{\cos(\phi Q)}{D}+3A\frac{\cos^2(\phi Q)}{D^2}-B=zero$$

$$-Q^2H\frac{\cos(\phi Q)}{D}+H\frac{\cos(\phi Q)}{D}-\frac{3Q^2A}{\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{6A}{\varepsilon D}\frac{\cos(\phi Q)}{D}-$$

$$-3Q^2A\frac{\cos^2(\phi Q)}{D^2}+3A\frac{\cos^2(\phi Q)}{D^2}+H\frac{1}{\varepsilon D}+\frac{3A}{\varepsilon^2D^2}-B=zero$$

$$\left(-Q^2H+H-\frac{3Q^2A}{\varepsilon D}+\frac{6A}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\left(-3Q^2A+3A\right)\frac{\cos^2(\phi Q)}{D^2}+H\frac{1}{\varepsilon D}+\frac{3A}{\varepsilon^2D^2}-B=zero$$

$$\left(-3Q^2A+3A\right)\frac{\cos^2(\phi Q)}{3AD^2}+\left(-Q^2H+H-\frac{3Q^2A}{\varepsilon D}+\frac{6A}{\varepsilon D}\right)\frac{\cos(\phi Q)}{3AD}+H\frac{1}{3A\varepsilon D}+\frac{3A}{3A\varepsilon^2D^2}-\frac{B}{3A}=zero$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}-\frac{B}{3A}=zero \qquad 21.40$$

$$Q^2\approx 1 \qquad \left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}=zero \qquad 21.41$$

$$\left(\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}-\frac{B}{3A}=zero \qquad 21.42$$

$$\frac{\cos(\phi Q)}{D}=zero\Rightarrow\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}-\frac{B}{3A}=zero$$

$$\frac{\cos(\phi Q)}{D}\neq zero\Rightarrow\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}=zero$$

$$\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}=zero \qquad \frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}-\frac{B}{3A}=zero \qquad 21.43$$

$$[a=b]\Rightarrow\frac{H}{3A}+\frac{1}{\varepsilon D}=\frac{1}{Q^2}\left(\frac{H}{3A}+\frac{2}{\varepsilon D}\right) \qquad [a=c]\Rightarrow\frac{H}{3A}+\frac{1}{\varepsilon D}=\frac{\varepsilon DB}{3A} \qquad 21.44$$

$$Q^2=1 \qquad H=\frac{E_R}{m_oc^2}=\frac{m_oc^2}{m_oc^2}=1 \qquad \varepsilon DB=\frac{\varepsilon DGM_o}{L^2}=\frac{\varepsilon DGM_o}{\varepsilon DGM_o}=1$$

$$[a=b]\Rightarrow\frac{H}{3A}+\frac{1}{\varepsilon D}=\frac{1}{1}\left(\frac{H}{3A}+\frac{2}{\varepsilon D}\right)\Rightarrow\frac{1}{\varepsilon D}=zero \qquad [a=c]\Rightarrow\frac{1}{3A}+\frac{1}{\varepsilon D}=\frac{1}{3A}\Rightarrow\frac{1}{\varepsilon D}=zero$$

$$[b=c]\Rightarrow\frac{1}{Q^2}\left(\frac{H}{3A}+\frac{2}{\varepsilon D}\right)=\frac{\varepsilon DB}{3A} \qquad 21.45$$

$$\varepsilon DB=\frac{\varepsilon DGM_o}{L^2}=\frac{\varepsilon DGM_o}{\varepsilon DGM_o}=1 \qquad 21.46$$

$$[b=c]\Rightarrow\frac{1}{Q^2}\left(\frac{H}{3A}+\frac{2}{\varepsilon D}\right)=\frac{1}{3A} \qquad Q^2=H+\frac{6A}{\varepsilon D} \qquad 21.47$$

$Q=Q(H)$ The regression is a function of positive energy that governs the movement.

$$H=\frac{E_R}{m_o c^2}=\frac{m_o c^2}{m_o c^2}=1 \qquad Q^2=1+\frac{6A}{\varepsilon D} \text{ Regression} \qquad 21.48$$

$$[a=b]\Rightarrow\frac{1}{3A}+\frac{1}{\varepsilon D}=\frac{1}{\left(1+\frac{6A}{\varepsilon D}\right)}\left(\frac{1}{3A}+\frac{2}{\varepsilon D}\right)\Rightarrow\frac{1}{\varepsilon D}=zero \qquad 21.49$$

$$3A\varepsilon D\left(\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)=zero \qquad 3A\varepsilon^2D^2\left(\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}-\frac{B}{3A}\right)=zero \qquad 21.43$$

$$H=\frac{E_R}{m_o c^2} \qquad A=\frac{GM_o}{c^2} \qquad B=\frac{GM_o}{L^2}$$

$$-Q^2H\varepsilon D+H\varepsilon D-Q^2 3A+6A=zero \qquad H\varepsilon D+3A-\varepsilon D(\varepsilon DB)=zero$$

$$-Q^2(-3A+\varepsilon D)-3A+\varepsilon D-Q^2 3A+6A=zero \qquad H\varepsilon D=-3A+\varepsilon D$$

$$Q^2 3A-Q^2\varepsilon D+\varepsilon D-Q^2 3A+3A=zero$$

$$-Q^2\varepsilon D+\varepsilon D+3A=zero \qquad Q^2=1+\frac{3A}{\varepsilon D}$$

This regression is not governed by the positive energy

$$\vec{v}=\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}} \tag{21.06}$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d}{dt}\left(\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}\right)=\frac{dt'}{dt}\frac{d}{dt'}\left(\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}\right)=\sqrt{1-\frac{v^2}{c^2}}\frac{d}{dt'}\left(\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}\right) \tag{21.50}$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\sqrt{1-\frac{v^2}{c^2}}\frac{-1}{\left(1+\frac{v'^2}{c^2}\right)}\left[\sqrt{1+\frac{v'^2}{c^2}}\frac{d\vec{v}'}{dt'}-\vec{v}'\frac{d}{dt'}\left(\sqrt{1+\frac{v'^2}{c^2}}\right)\right]$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\sqrt{1-\frac{v^2}{c^2}}\frac{-1}{\left(1+\frac{v'^2}{c^2}\right)}\left[\sqrt{1+\frac{v'^2}{c^2}}\frac{d\vec{v}'}{dt'}-\vec{v}'\frac{1}{2}\left(1+\frac{v'^2}{c^2}\right)^{\frac{1}{2}-\frac{2}{2}=\frac{-1}{2}}\left(\frac{2v'}{c^2}\frac{dv'}{dt'}\right)\right]$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\sqrt{1-\frac{v^2}{c^2}}\frac{-1}{\left(1+\frac{v'^2}{c^2}\right)}\left(\sqrt{1+\frac{v'^2}{c^2}}\frac{d\vec{v}'}{dt'}-\frac{1}{\sqrt{1+\frac{v'^2}{c^2}}}v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right)$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\sqrt{1-\frac{v^2}{c^2}}\frac{-1}{\left(1+\frac{v'^2}{c^2}\right)}\left(\frac{1}{\sqrt{1+\frac{v'^2}{c^2}}}\sqrt{1+\frac{v'^2}{c^2}}\frac{d\vec{v}'}{dt'}\sqrt{1+\frac{v'^2}{c^2}}-\frac{1}{\sqrt{1+\frac{v'^2}{c^2}}}v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right)$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\sqrt{1-\frac{v^2}{c^2}}\frac{-1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'}-v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right]$$

$$m\vec{a}=\frac{m_0\vec{a}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt}=\frac{-m_0}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'}-v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right]$$

$$\vec{F}=m\vec{a}=\frac{m_0\vec{a}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt} \tag{21.51}$$

$$\vec{F}'=\frac{-m_0}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'}-v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right] \tag{21.52}$$

$$\vec{F}=m\vec{a}=\frac{m_0\vec{a}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt}=\vec{F}'=\frac{-m_0}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'}-v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right] \tag{21.53}$$

$$E_k=\int \vec{F}\cdot d\vec{r}=\int \vec{F}'\cdot\left(-d\vec{r}'\right)=\int \frac{-k}{r^2}\hat{r}\left(-d\vec{r}'\right) \quad 21.54$$

$$E_k=\int \vec{F}\cdot d\vec{r}=\int \vec{F}'\cdot\left(-d\vec{r}'\right)=\int \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt}d\vec{r}=\int \frac{-m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'}-v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right]\left(-d\vec{r}'\right)=\int \frac{-k}{r^2}\hat{r}\left(-d\vec{r}'\right) \quad 21.55$$

$$E_k=\int \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}d\vec{v}\frac{d\vec{r}}{dt}=\int \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)d\vec{v}'\frac{d\vec{r}'}{dt'}-v'dv'\frac{d\vec{r}'}{dt'}\frac{\vec{v}'}{c^2}\right]=\int \frac{k}{r^2}\hat{r}\,d\vec{r}'$$

$$E_k=\int \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}d\vec{v}\vec{v}=\int \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)d\vec{v}'\vec{v}'-v'dv'\frac{\vec{v}'\vec{v}'}{c^2}\right]=\int \frac{-k}{r^2}dr$$

$$E_k=\int \frac{m_o v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}=\int \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)d\vec{v}'\vec{v}'-v'dv'\frac{v'^2}{c^2}\right]=\int \frac{-k}{r^2}dr$$

$$E_k=\int \frac{m_o v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}=\int \frac{m_o v'dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left(1+\frac{v'^2}{c^2}-\frac{v'^2}{c^2}\right)=\int \frac{-k}{r^2}dr$$

$$E_k=\int \frac{m_o v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}=\int \frac{m_o v'dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=\int \frac{-k}{r^2}dr \qquad dE_k=\frac{m_o v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_o v'dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=\frac{-k}{r^2}dr \quad 21.56$$

$$E_k=-m_oc^2\sqrt{1-\frac{v^2}{c^2}}=\frac{-m_oc^2}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{k}{r}+constant \quad 21.57$$

$$E_R=-m_oc^2\sqrt{1-\frac{v^2}{c^2}}-\frac{k}{r}=constant \qquad E_R=\frac{-m_oc^2}{\sqrt{1+\frac{v'^2}{c^2}}}-\frac{k}{r}=constant \quad 21.58$$

$$E_R=\frac{-m_oc^2}{\sqrt{1+\frac{v'^2}{c^2}}}-\frac{k}{r}=-m_oc^2+\frac{m_ov'^2}{2}-\frac{k}{r} \qquad E_R=\frac{-m_oc^2}{\sqrt{1+\frac{(0)^2}{c^2}}}-\frac{k}{\infty}=-m_oc^2 \quad 21.59$$

$$\frac{-1}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{E_R}{m_oc^2}+\frac{k}{m_oc^2}\frac{1}{r} \quad 21.60$$

$$H=\frac{E_R}{m_oc^2} \qquad A=\frac{k}{m_oc^2}=\frac{GMm_o}{m_oc^2}=\frac{GM_o}{c^2} \quad 21.61$$

$$\frac{-1}{\sqrt{1+\frac{v'^2}{c^2}}}=H+A\frac{1}{r} \qquad \frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=-\left(H+A\frac{1}{r}\right)^3 \qquad 21.62$$

$$\vec{L}'=\vec{r}'\times\vec{v}'=-r\hat{r}\times\left[-\left(\frac{dr}{dt'}\hat{r}+r\frac{d\phi}{dt'}\hat{\phi}\right)\right]=r^2\frac{d\phi}{dt'}\left(\hat{r}\times\hat{\phi}\right)=r^2\frac{d\phi}{dt'}\hat{k} \qquad 21.63$$

$$\vec{L}'=\vec{r}'x\vec{v}'=-r\hat{r}\times\frac{-\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}=r\hat{r}\times\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[\left(\frac{dr}{dt}\hat{r}+r\frac{d\phi}{dt}\hat{\phi}\right)\right]=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}r^2\frac{d\phi}{dt}\left(\hat{r}\times\hat{\phi}\right)=r^2\frac{d\phi}{dt'}\hat{k} \qquad 21.63$$

$$\vec{L}'=r^2\frac{d\phi}{dt'}\hat{k}=L'\hat{k} \qquad L'=r^2\frac{d\phi}{dt'} \qquad 21.64$$

$$dE_k=\frac{m_o v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_o v'dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=\frac{-k}{r^2}dr=\frac{k}{r^2}\hat{r}d\vec{r}' \qquad 21.56$$

$$\frac{dE_k}{dt'}=\vec{F}'\vec{v}'=\frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}v'\frac{dv'}{dt'}=\frac{k}{r^2}\hat{r}\frac{d\vec{r}'}{dt'}=\frac{k}{r^2}\hat{r}\vec{v}'$$

$$\vec{F}'=\frac{m_o\vec{a}'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=\frac{k}{r^2}\hat{r} \qquad 21.65$$

$$\vec{F}'=\frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left\{-\left[\frac{d^2r}{dt'^2}-r\left(\frac{d\phi}{dt'}\right)^2\right]\hat{r}-\left(2\frac{dr}{dt'}\frac{d\phi}{dt'}+r\frac{d^2\phi}{dt'^2}\right)\hat{\phi}\right\}=\frac{k}{r^2}\hat{r} \qquad 21.66$$

$$F'_{\hat{\phi}}=\frac{-m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left(2\frac{dr}{dt'}\frac{d\phi}{dt'}+r\frac{d^2\phi}{dt'^2}\right)\hat{\phi}=zero \qquad 21.67$$

$$F'_{\hat{r}}=\frac{-m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{d^2r}{dt'^2}-r\left(\frac{d\phi}{dt'}\right)^2\right]\hat{r}=\frac{k}{r^2}\hat{r} \qquad 21.68$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{d^2r}{dt'^2}-r\left(\frac{d\phi}{dt'}\right)^2\right]\hat{r}=\frac{-GM_o}{r^2}\hat{r}$$

$$\frac{d\phi}{dt'}=\frac{L'}{r^2} \qquad \frac{dr}{dt'}=-L'\frac{dw}{d\phi} \qquad \frac{d^2r}{dt'^2}=\frac{-L'^2}{r^2}\frac{d^2w}{d\phi^2} \qquad \frac{d^2\phi}{dt'^2}=\frac{2L'^2}{r^3}\frac{dw}{d\phi} \qquad 21.69$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{-L'^2}{r^2}\frac{d^2w}{d\phi^2}-r\left(\frac{L'}{r^2}\right)^2\right]=\frac{-GM_\odot}{r^2}$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left(\frac{-L'^2}{r^2}\frac{d^2w}{d\phi^2}-\frac{L'^2}{r^3}\right)=\frac{-GM_\odot}{r^2}$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)\left(\frac{-L'^2}{r^2}\right)=\frac{-GM_\odot}{r^2}$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{GM_\odot}{L'^2} \qquad 21.70$$

$$-\left(H+A\frac{1}{r}\right)^3\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{GM_\odot}{L'^2} \qquad 21.71$$

$$\left(H+3A\frac{1}{r}\right)\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{-GM_\odot}{L'^2}$$

$$H\frac{d^2w}{d\phi^2}+H\frac{1}{r}+3A\frac{d^2w}{d\phi^2}\frac{1}{r}+3A\frac{1}{r^2}=\frac{-GM_\odot}{L'^2}$$

$$H\frac{d^2w}{d\phi^2}+Hw+3A\frac{d^2w}{d\phi^2}w+3Aw^2+\frac{GM_\odot}{L'^2}=zero$$

$$H=\frac{E_R}{m_\odot c^2} \qquad A=\frac{GM_\odot}{c^2} \qquad B=\frac{GM_\odot}{L'^2} \qquad 21.72$$

$$H\frac{d^2w}{d\phi^2}+Hw+3A\frac{d^2w}{d\phi^2}w+3Aw^2+B=zero \qquad 21.73$$

$$w=\frac{1}{r}=\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)] \qquad \frac{dw}{d\phi}=\frac{-Q\,sen(\phi Q)}{D} \qquad \frac{d^2w}{d\phi^2}=\frac{-Q^2\cos(\phi Q)}{D} \qquad 21.38$$

$$H\frac{-Q^2\cos(\phi Q)}{D}+H\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]+3A\frac{-Q^2\cos(\phi Q)}{D}\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]+3A\left\{\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]\right\}^2+B=zero \qquad 21.74$$

$$-Q^2H\frac{\cos(\phi Q)}{D}+H\frac{1}{\varepsilon D}+H\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q)-\frac{3Q^2A}{\varepsilon D}\frac{\cos(\phi Q)}{D}[1+\varepsilon\cos(\phi Q)]+\frac{3A}{\varepsilon^2D^2}\left[1+2\varepsilon\cos(\phi Q)+\varepsilon^2\cos^2(\phi Q)\right]+B=zero$$

$$-Q^2H\frac{\cos(\phi Q)}{D}+H\frac{1}{\varepsilon D}+H\frac{\cos(\phi Q)}{D}-\frac{3Q^2A}{\varepsilon D}\frac{\cos(\phi Q)}{D}-\frac{3Q^2A}{\varepsilon D}\frac{\cos(\phi Q)}{D}\varepsilon\cos(\phi Q)+$$
$$+\frac{3A}{\varepsilon^2D^2}+\frac{3A}{\varepsilon^2D^2}2\varepsilon\cos(\phi Q)+\frac{3A}{\varepsilon^2D^2}\varepsilon^2\cos^2(\phi Q)+B=zero$$

$$-Q^2H\frac{\cos(\phi Q)}{D}+H\frac{1}{\varepsilon D}+H\frac{\cos(\phi Q)}{D}-\frac{3Q^2A}{\varepsilon D}\frac{\cos(\phi Q)}{D}-3Q^2A\frac{\cos^2(\phi Q)}{D^2}+$$

$$+\frac{3A}{\varepsilon^2D^2}+\frac{6A}{\varepsilon D}\frac{\cos(\phi Q)}{D}+3A\frac{\cos^2(\phi Q)}{D^2}+B=zero$$

$$-Q^2H\frac{\cos(\phi Q)}{D}+H\frac{\cos(\phi Q)}{D}-\frac{3Q^2A}{\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{6A}{\varepsilon D}\frac{\cos(\phi Q)}{D}-$$

$$-3Q^2A\frac{\cos^2(\phi Q)}{D^2}+3A\frac{\cos^2(\phi Q)}{D^2}+H\frac{1}{\varepsilon D}+\frac{3A}{\varepsilon^2D^2}+B=zero$$

$$\left(-Q^2H+H-\frac{3Q^2A}{\varepsilon D}+\frac{6A}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\left(-3Q^2A+3A\right)\frac{\cos^2(\phi Q)}{D^2}+H\frac{1}{\varepsilon D}+\frac{3A}{\varepsilon^2D^2}+B=zero$$

$$\left(-3Q^2A+3A\right)\frac{\cos^2(\phi Q)}{3AD^2}+\left(-Q^2H+H-\frac{3Q^2A}{\varepsilon D}+\frac{6A}{\varepsilon D}\right)\frac{\cos(\phi Q)}{3AD}+H\frac{1}{3A\varepsilon D}+\frac{3A}{3A\varepsilon^2D^2}+\frac{B}{3A}=zero$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3A}=zero \qquad 21.75$$

$$Q^2\approx 1 \qquad \left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}=zero \qquad 21.76$$

$$\left(\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3A}=zero \qquad 21.77$$

$$\frac{\cos(\phi Q)}{D}=zero\Rightarrow\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3A}=zero$$

$$\frac{\cos(\phi Q)}{D}\neq zero\Rightarrow\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}=zero$$

$$\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}=zero \qquad \frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3A}=zero \qquad 21.78$$

$$[a=b]\Rightarrow\frac{H}{3A}+\frac{1}{\varepsilon D}=\frac{1}{Q^2}\left(\frac{H}{3A}+\frac{2}{\varepsilon D}\right) \qquad [a=c]\Rightarrow\frac{H}{3A}+\frac{1}{\varepsilon D}=-\frac{\varepsilon DB}{3A} \qquad 21.79$$

$$Q^2=1 \qquad H=\frac{E_R}{m_0c^2}=\frac{-m_0c^2}{m_0c^2}=-1 \qquad \varepsilon DB=\frac{\varepsilon DGM_0}{L'^2}=\frac{\varepsilon DGM_0}{\varepsilon DGM_0}=1$$

$$[a=b]\Rightarrow\frac{H}{3A}+\frac{1}{\varepsilon D}=\frac{1}{1}\left(\frac{H}{3A}+\frac{2}{\varepsilon D}\right)\Rightarrow\frac{1}{\varepsilon D}=zero \qquad [a=c]\Rightarrow\frac{-1}{3A}+\frac{1}{\varepsilon D}=-\frac{1}{3A}\Rightarrow\frac{1}{\varepsilon D}=zero$$

$$[b=c]\Rightarrow\frac{1}{Q^2}\left(\frac{H}{3A}+\frac{2}{\varepsilon D}\right)=-\frac{\varepsilon DB}{3A} \qquad 21.80$$

$$\varepsilon DB=\frac{\varepsilon DGM_0}{L'^2}=\frac{\varepsilon DGM_0}{\varepsilon DGM_0}=1 \qquad 21.81$$

$$[b=c] \Rightarrow \frac{1}{Q^2}\left(\frac{H}{3A}+\frac{2}{\varepsilon D}\right)=-\frac{1}{3A} \qquad Q^2=-H-\frac{6A}{\varepsilon D} \qquad 21.82$$

$Q=Q(H)$ The advance is a function of negative energy that governs the movement

$$H=\frac{E_R}{m_0c^2}=\frac{-m_0c^2}{m_0c^2}=-1 \qquad Q^2=-(-1)-\frac{6A}{\varepsilon D} \Rightarrow Q^2=1-\frac{6A}{\varepsilon D} \text{ Advance} \qquad 21.83$$

$$[a=b] \Rightarrow \frac{-1}{3A}+\frac{1}{\varepsilon D}=\frac{1}{\left(1-\frac{6A}{\varepsilon D}\right)}\left(\frac{-1}{3A}+\frac{2}{\varepsilon D}\right) \Rightarrow \frac{1}{\varepsilon D}=zero \qquad 21.84$$

$$H=\frac{E_R}{m_0c^2} \qquad A=\frac{GM_0}{c^2} \qquad B=\frac{GM_0}{L'^2}$$

$$\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}=zero \qquad \frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3A}=zero \qquad 21.78$$

$$3A\varepsilon D\left(\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)=zero \qquad 3A\varepsilon^2D^2\left(\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3A}\right)=zero$$

$$-Q^2H\varepsilon D+H\varepsilon D-Q^23A+6A=zero \qquad H\varepsilon D+3A+\varepsilon D(\varepsilon DB)=zero \qquad 21.85$$

$$\varepsilon DB=\frac{\varepsilon DGM_0}{L'^2}=\frac{\varepsilon DGM_0}{\varepsilon DGM_0}=1 \qquad H\varepsilon D=-3A-\varepsilon D \qquad 21.86$$

$$-Q^2(-3A-\varepsilon D)-3A-\varepsilon D-Q^23A+6A=zero \qquad 21.87$$

$$Q^23A+Q^2\varepsilon D-\varepsilon D-Q^23A+3A=zero$$

$$Q^2\varepsilon D-\varepsilon D+3A=zero \qquad Q^2=1-\frac{3A}{\varepsilon D} \qquad 21.88$$

This advance is not governed by negative energy

$$-Q^2H\varepsilon D+H\varepsilon D-Q^23A+6A=zero \qquad 21.85$$

$$-Q^2(-3A-\varepsilon D)+H\varepsilon D-Q^23A+6A=zero \qquad 21.89$$

$$Q^23A+Q^2\varepsilon D+H\varepsilon D-Q^23A+6A=zero$$

$$Q^2\varepsilon D+H\varepsilon D+6A=zero \qquad Q^2=-H-\frac{6A}{\varepsilon D} \qquad 21.90$$

$$\left(\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3A}=zero \qquad 21.77$$

$$3A\varepsilon^2D^2\left[\left(\frac{-Q^2H}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3A}\right]=zero$$

$$\varepsilon D\left(\frac{-Q^2H3A\varepsilon D}{3A}+\frac{H3A\varepsilon D}{3A}-\frac{Q^23A\varepsilon D}{\varepsilon D}+\frac{2.3A\varepsilon D}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{H3A\varepsilon^2D^2}{3A\varepsilon D}+\frac{3A\varepsilon^2D^2}{\varepsilon^2D^2}+\frac{B3A\varepsilon^2D^2}{3A}=zero$$

$$\varepsilon D\left(-Q^2 H\varepsilon D+H\varepsilon D-Q^2 3A+6A\right)\frac{\cos(\phi Q)}{D}+H\varepsilon D+3A+\varepsilon D(\varepsilon DB)=zero$$

$$\varepsilon DB=\frac{\varepsilon DGM_o}{L^2}=\frac{\varepsilon DGM_o}{\varepsilon DGM_o}=1 \qquad H=\frac{E_R}{m_o c^2}=\frac{-m_o c^2}{m_o c^2}=-1$$

$$\varepsilon D\left(-Q^2 H\varepsilon D+H\varepsilon D-Q^2 3A+6A\right)\frac{\cos(\phi Q)}{D}-\varepsilon D+3A+\varepsilon D=zero$$

$$\left(-Q^2 H\varepsilon D+H\varepsilon D-Q^2 3A+6A\right)\frac{\cos(\phi Q)}{D}+\frac{3A}{\varepsilon D}=zero \qquad 21.91$$

$$Q^2=1-\frac{3A}{\varepsilon D}$$

$$\left[-\left(1-\frac{3A}{\varepsilon D}\right)H\varepsilon D+H\varepsilon D-\left(1-\frac{3A}{\varepsilon D}\right)3A+6A\right]\frac{\cos(\phi Q)}{D}+\frac{3A}{\varepsilon D}=zero$$

$$\left(-H\varepsilon D+H\varepsilon D\frac{3A}{\varepsilon D}+H\varepsilon D-3A+3A\frac{3A}{\varepsilon D}+6A\right)\frac{\cos(\phi Q)}{D}+\frac{3A}{\varepsilon D}=zero$$

$$\left(-H\varepsilon D+H3A+H\varepsilon D-3A+\frac{9A^2}{\varepsilon D}+6A\right)\frac{\cos(\phi Q)}{D}+\frac{3A}{\varepsilon D}=zero$$

$$\left(H3A+\frac{9A^2}{\varepsilon D}+3A\right)\frac{\cos(\phi Q)}{D}+\frac{3A}{\varepsilon D}=zero$$

$$H=\frac{E_R}{m_o c^2}=\frac{-m_o c^2}{m_o c^2}=-1$$

$$\left(-3A+\frac{9A^2}{\varepsilon D}+3A\right)\frac{\cos(\phi Q)}{D}+\frac{3A}{\varepsilon D}=zero$$

$$\frac{9A^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{3A}{\varepsilon D}=zero \qquad \frac{\cos(\phi Q)}{D}+\frac{1}{3A}=zero \qquad 21.92$$

$$\left(-Q^2 H\varepsilon D+H\varepsilon D-Q^2 3A+6A\right)\frac{\cos(\phi Q)}{D}+\frac{3A}{\varepsilon D}=zero \qquad 21.91$$

$$Q^2=1-\frac{6A}{\varepsilon D}$$

$$\left[-\left(1-\frac{6A}{\varepsilon D}\right)H\varepsilon D+H\varepsilon D-\left(1-\frac{6A}{\varepsilon D}\right)3A+6A\right]\frac{\cos(\phi Q)}{D}+\frac{3A}{\varepsilon D}=zero$$

$$\left(-H\varepsilon D+H\varepsilon D\frac{6A}{\varepsilon D}+H\varepsilon D-3A+3A\frac{6A}{\varepsilon D}+6A\right)\frac{\cos(\phi Q)}{D}+\frac{3A}{\varepsilon D}=zero$$

$$\left(-H\varepsilon D+H6A+H\varepsilon D-3A+\frac{18A^2}{\varepsilon D}+6A\right)\frac{\cos(\phi Q)}{D}+\frac{3A}{\varepsilon D}=zero$$

$$\left(H6A+\frac{18A^2}{\varepsilon D}+3A\right)\frac{\cos(\psi/Q)}{D}+\frac{3A}{\varepsilon D}=zero$$

$$H=\frac{E_R}{m_o c^2}=\frac{-m_o c^2}{m_o c^2}=-1$$

$$\left(-6A+\frac{18A^2}{\varepsilon D}+3A\right)\frac{\cos(\psi/Q)}{D}+\frac{3A}{\varepsilon D}=zero$$

$$\frac{1}{3A}\left[\left(-3A+\frac{18A^2}{\varepsilon D}\right)\frac{\cos(\psi/Q)}{D}+\frac{3A}{\varepsilon D}\right]=zero$$

$$\left(-1+\frac{6A}{\varepsilon D}\right)\frac{\cos(\psi/Q)}{D}+\frac{1}{\varepsilon D}=zero$$

$$-\left(1-\frac{6A}{\varepsilon D}\right)\frac{\cos(\psi/Q)}{D}+\frac{1}{\varepsilon D}=zero \qquad -Q^2\frac{\cos(\psi/Q)}{D}+\frac{1}{\varepsilon D}=zero \qquad 21.93$$

$$\left(-Q^2 H\varepsilon D+H\varepsilon D-Q^2 3A+6A\right)\frac{\cos(\psi/Q)}{D}+\frac{3A}{\varepsilon D}=zero \qquad 21.91$$

$$Q^2=1 \qquad H=\frac{E_R}{m_o c^2}=\frac{-m_o c^2}{m_o c^2}=-1$$

$$(\varepsilon D-\varepsilon D-3A+6A)\frac{\cos(\psi/Q)}{D}+\frac{3A}{\varepsilon D}=zero$$

$$(3A)\frac{\cos(\psi/Q)}{D}+\frac{3A}{\varepsilon D}=zero \qquad \frac{\cos(\psi/Q)}{D}+\frac{1}{\varepsilon D}=zero \qquad 21.94$$

$$Q^2=1-\frac{6A}{\varepsilon D} \qquad Q^2=1 \qquad Q^2=1-\frac{3A}{\varepsilon D}$$

$$\left|-Q^2\frac{\cos(\psi/Q)}{D}+\frac{1}{\varepsilon D}\right| << \left|\frac{\cos(\psi/Q)}{D}+\frac{1}{\varepsilon D}\right| <<<<<< \left|\frac{\cos(\psi/Q)}{D}+\frac{1}{3A}\right| \qquad 21.95$$

$$E_N=\frac{m_o u^2}{2}-\frac{k}{r}$$

$$u^2=\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\phi}{dt}\right)^2=\left(\frac{dr}{dt}\right)^2+\frac{L^2}{r^2}$$

$$E_N=\frac{m_o}{2}\left[\left(\frac{dr}{dt}\right)^2+\frac{L^2}{r^2}\right]-\frac{k}{r}$$

$$\frac{2E_N}{m_o}=\left(\frac{dr}{dt}\right)^2+\frac{L^2}{r^2}-\frac{2k}{m_o}\frac{1}{r}$$

$$\left(\frac{dr}{dt}\right)^2+\frac{L^2}{r^2}-\frac{2k}{m_o}\frac{1}{r}-\frac{2E_N}{m_o}=zero$$

$$\frac{d\phi}{dt}=\frac{L}{r^2}\qquad \frac{dr}{dt}=-L\frac{dw}{d\phi}\qquad \frac{d^2r}{dt^2}=\frac{-L^2}{r^2}\frac{d^2w}{d\phi^2}\qquad \frac{d^2\phi}{dt^2}=\frac{2L^2}{r^3}\frac{dw}{d\phi}$$

$$\left(-L\frac{dw}{d\phi}\right)^2+\frac{L^2}{r^2}-\frac{2k}{m_o}\frac{1}{r}-\frac{2E_N}{m_o}=zero$$

$$\left(\frac{dw}{d\phi}\right)^2+\frac{1}{r^2}-\frac{2k}{m_oL^2}\frac{1}{r}-\frac{2E_N}{m_oL^2}=zero$$

$$\left(\frac{dw}{d\phi}\right)^2+\frac{1}{r^2}-\frac{2k}{m_oL^2}\frac{1}{r}-\frac{2E_N}{m_oL^2}=zero$$

$$\left(\frac{dw}{d\phi}\right)^2+w^2-\frac{2k}{m_oL^2}w-\frac{2E_N}{m_oL^2}=zero$$

$$x=\frac{2k}{m_oL^2}\qquad y=\frac{2E_N}{m_oL^2}$$

$$\left(\frac{dw}{d\phi}\right)^2+w^2-xw-y=zero$$

$$w=\frac{1}{r}=\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right]\qquad \frac{dw}{d\phi}=\frac{-Q\,sen(\phi Q)}{D}\qquad \frac{d^2w}{d\phi^2}=\frac{-Q^2\cos(\phi Q)}{D}$$

$$\left[\frac{-Q\,sen(\phi Q)}{D}\right]^2+\left\{\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right]\right\}^2-x\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right]-y=zero$$

$$\frac{Q^2}{D^2}\left[1-\cos^2(\phi Q)\right]+\frac{1}{\varepsilon^2D^2}\left[1+2\varepsilon\cos(\phi Q)+\varepsilon^2\cos^2(\phi Q)\right]-x\frac{1}{\varepsilon D}-x\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q)-y=zero$$

$$\frac{Q^2}{D^2}-\frac{Q^2}{D^2}\cos^2(\phi Q)+\frac{1}{\varepsilon^2 D^2}+\frac{1}{\varepsilon^2 D^2}2\varepsilon\cos(\phi Q)+\frac{1}{\varepsilon^2 D^2}\varepsilon^2\cos^2(\phi Q)-\frac{x}{\varepsilon D}-x\frac{\cos(\phi Q)}{D}-y=zero$$

$$\frac{Q^2}{D^2}-Q^2\frac{\cos^2(\phi Q)}{D^2}+\frac{1}{\varepsilon^2 D^2}+\frac{2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{\cos^2(\phi Q)}{D^2}-\frac{x}{\varepsilon D}-x\frac{\cos(\phi Q)}{D}-y=zero$$

$$\frac{\cos^2(\phi Q)}{D^2}-Q^2\frac{\cos^2(\phi Q)}{D^2}+\frac{2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-x\frac{\cos(\phi Q)}{D}+\frac{Q^2}{D^2}+\frac{1}{\varepsilon^2 D^2}-\frac{x}{\varepsilon D}-y=zero$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{2}{\varepsilon D}-x\right)\frac{\cos(\phi Q)}{D}+\frac{Q^2}{D^2}+\frac{1}{\varepsilon^2 D^2}-\frac{x}{\varepsilon D}-y=zero$$

$$Q^2\approx 1 \qquad \left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}=zero$$

$$\left(\frac{2}{\varepsilon D}-x\right)\frac{\cos(\phi Q)}{D}+\frac{1}{D^2}+\frac{1}{\varepsilon^2 D^2}-\frac{x}{\varepsilon D}-y=zero$$

$$\left(\frac{2}{\varepsilon D}-x\right)=zero \qquad \frac{1}{D^2}+\frac{1}{\varepsilon^2 D^2}-\frac{x}{\varepsilon D}-y=zero$$

$$x=\frac{2k}{m_o L^2} \qquad y=\frac{2E_N}{m_o L^2}$$

$$\frac{2}{\varepsilon D}-x=zero\Rightarrow x=\frac{2}{\varepsilon D}=\frac{2k}{m_o L^2}\Rightarrow\frac{1}{\varepsilon D}=\frac{GM_o m_o}{m_o L^2}\Rightarrow L^2=\varepsilon DGM_o$$

$$\frac{\varepsilon^2 D^2}{D^2}+\frac{\varepsilon^2 D^2}{\varepsilon^2 D^2}-\frac{\varepsilon^2 D^2 x}{\varepsilon D}-\varepsilon^2 D^2 y=zero$$

$$\varepsilon^2+1-\varepsilon Dx-\varepsilon^2 D^2 y=zero$$

$$\varepsilon Dx=\varepsilon D\frac{2}{\varepsilon D}\Rightarrow\varepsilon Dx=2 \qquad \varepsilon^2 D^2 y=\varepsilon^2 D^2\frac{2E_N}{m_o L^2}=\varepsilon^2 D^2\frac{2E_N}{m_o\varepsilon DGM_o}=\frac{2\varepsilon DE_N}{k}$$

$$\varepsilon^2+1-2-\frac{2\varepsilon DE_N}{k}=zero \qquad E_N=\frac{k}{2\varepsilon D}\left(\varepsilon^2-1\right)$$

$$\frac{1}{a}=\frac{1}{\varepsilon D}\left(1-\varepsilon^2\right) \qquad E_N=\frac{-k}{2a}$$

§22 Spatial deformation

$$t = \frac{t'}{\sqrt{1-\frac{v^2}{c^2}}} \qquad t > t'$$

$$t = t_1 + t_2 = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2L}{c}\frac{1}{\left(1-\frac{v^2}{c^2}\right)} \qquad t' = \frac{2L'}{c}$$

$$t = \frac{2L}{c}\frac{1}{\left(1-\frac{v^2}{c^2}\right)} = \frac{\frac{2L'}{c}}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow L = L'\sqrt{1-\frac{v^2}{c^2}} \qquad L' > L$$

This is the spatial deformation.

The length L' at rest in the reference frame of the observer O' is greater than the length L that is moving with velocity relative v on reference frame the observer O.

Now compute to the observer O' the distance $d' = vt'$ between $O \leftrightarrow O'$:

$$d' = vt' = v\frac{2L'}{c}$$

Thus we obtain the velocity v: $d' = v\frac{2L'}{c} \Rightarrow v = \frac{cd'}{2L'}$.

Now compute to the observer O the distance $d = vt$ between $O \leftrightarrow O'$:

$$d = vt = v(t_1 + t_2) = v\frac{2L}{c}\frac{1}{\left(1-\frac{v^2}{c^2}\right)}$$

Thus we obtain the velocity v: $d = v\frac{2L}{c}\frac{1}{\left(1-\frac{v^2}{c^2}\right)} \Rightarrow v = \frac{cd}{2L}\left(1-\frac{v^2}{c^2}\right)$.

The speed v is the same to both observers so we have:

$$v = \frac{cd'}{2L'} = \frac{cd}{2L}\left(1-\frac{v^2}{c^2}\right)$$

Where applying the relation $L = L'\sqrt{1-\frac{v^2}{c^2}}$ we obtain:

$$\frac{cd'}{2L'} = \frac{cd}{2L'\sqrt{1-\frac{v^2}{c^2}}}\left(1-\frac{v^2}{c^2}\right) \Rightarrow d' = d\sqrt{1-\frac{v^2}{c^2}} \qquad d > d'.$$

Where the distance d and d’ varies inversely with the distances L and L’.

In general, we obtain (14.2, 14.4):

$$d' = \frac{d\left(1 - \frac{v u_x}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{or} \qquad d = \frac{d'\left(1 + \frac{v u'_x}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$u'_x = \text{zero} \qquad d = \frac{d'\left[1 + \frac{v(0)}{c^2}\right]}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad d = \frac{d'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$u'_x = c \qquad d = \frac{d'\left(1 + \frac{vc}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad d = d'\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

$$u'_x = -v \qquad d = \frac{d'\left[1 + \frac{v(-v)}{c^2}\right]}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad d = d'\sqrt{1 - \frac{v^2}{c^2}}$$

$$u_x = v \qquad d' = \frac{d\left[1 - \frac{v(v)}{c^2}\right]}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad d' = d\sqrt{1 - \frac{v^2}{c^2}}$$

$$u_x = c \qquad d' = \frac{d\left(1 - \frac{vc}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad d' = d\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$$

$$u_x = \text{zero} \qquad d' = \frac{d\left[1 - \frac{v(0)}{c^2}\right]}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad d' = \frac{d}{\sqrt{1 - \frac{v^2}{c^2}}}$$

§23 Space and Time Bend

Variables with line $t', v', x', y', \vec{r}'$ etc ...They are used in §21.

Geometry of space and time in the plan $xy \rightarrow y \perp x$.

$$y = f(x)$$

$$x = ct' \qquad y = \int ds' = \int \sqrt{d\vec{r}' \cdot d\vec{r}'}$$

$$\int ds' = f(ct')$$

$$dx = c\,dt' \qquad dy = ds' = \sqrt{d\vec{r}' \cdot d\vec{r}'}$$

$$\vec{r} = x\hat{i} + y\hat{j} = ct'\hat{i} + \int ds'\hat{j} \qquad \vec{r}' = x'\hat{i} + y'\hat{j}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j} = c\,dt'\hat{i} + ds'\hat{j} \qquad d\vec{r}' = dx'\hat{i} + dy'\hat{j}$$

$$dr = \frac{\vec{r} \cdot d\vec{r}}{r} = \frac{x}{r}dx + \frac{y}{r}dy$$

$$\vec{v} = \frac{d\vec{r}}{dt'} = \frac{dx}{dt'}\hat{i} + \frac{dy}{dt'}\hat{j} = \frac{c\,dt'}{dt'}\hat{i} + \frac{ds'}{dt'}\hat{j} = c\hat{i} + v'\hat{j} \qquad \vec{v}' = \frac{d\vec{r}'}{dt'} = \frac{dx'}{dt'}\hat{i} + \frac{dy'}{dt'}\hat{j}$$

$$\frac{dx}{dt'} = c \qquad \frac{dy}{dt'} = \frac{ds'}{dt'} = v' \qquad c = v\cos\varphi \qquad v' = v\,\mathrm{sen}\,\varphi$$

$$tg\varphi = \frac{dy}{dx} = \frac{\frac{dy}{dt'}}{\frac{dx}{dt'}} = \frac{\frac{ds'}{dt'}}{c} = \frac{1}{c}\frac{ds'}{dt'} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{c}\frac{d}{dt'}\left(\frac{1}{c}\frac{ds'}{dt'}\right) = \frac{1}{c^2}\frac{d^2s'}{dt'^2}$$

$$\vec{v} = \vec{c} + \vec{v}' \qquad \vec{c} = c\hat{i} \qquad \vec{v}' = v'\hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt'} = \frac{d\vec{c}}{dt'} + \frac{d\vec{v}'}{dt'} \qquad \frac{d\vec{c}}{dt'} = zero \qquad \frac{d\vec{v}}{dt'} = \frac{d\vec{v}'}{dt'} \rightarrow \vec{a} = \vec{a}'$$

$$ds^2 = d\vec{r} \cdot d\vec{r} = (dx\hat{i} + dy\hat{j})(dx\hat{i} + dy\hat{j}) = (c\,dt'\hat{i} + ds'\hat{j})(c\,dt'\hat{i} + ds'\hat{j}) = dx^2 + dy^2 = c^2dt'^2 + ds'^2$$

$$ds = \sqrt{c^2dt'^2 + ds'^2} \qquad ds' = \sqrt{ds^2 - c^2dt'^2}$$

$$v = \frac{ds}{dt'} = \sqrt{c^2 + \left(\frac{ds'}{dt'}\right)^2} = \sqrt{c^2 + v'^2} > c \qquad v' = \frac{ds'}{dt'} = \sqrt{\left(\frac{ds}{dt'}\right)^2 - c^2} = \sqrt{v^2 - c^2}$$

$$K=\left|\frac{d\varphi}{ds}\right| \to \varphi \equiv \angle$$ theoretical curve

$$tg\varphi=\frac{dy}{dx} \qquad \varphi=arctg\frac{dy}{dx} \qquad \frac{d\varphi}{dx}=\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}=\frac{\frac{1}{c^2}\frac{d^2s'}{dt'^2}}{1+\frac{1}{c^2}\left(\frac{ds'}{dt'}\right)^2}$$

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{c^2}\left(\frac{ds'}{dt'}\right)^2}$$

$$K=\frac{d\varphi}{ds}=\frac{\frac{d\varphi}{dx}}{\frac{ds}{dx}}=\frac{\frac{\frac{1}{c^2}\frac{d^2s'}{dt'^2}}{1+\frac{1}{c^2}\left(\frac{ds'}{dt'}\right)^2}}{\sqrt{1+\frac{1}{c^2}\left(\frac{ds'}{dt'}\right)^2}}=\frac{\frac{1}{c^2}\frac{d^2s'}{dt'^2}}{\left[1+\frac{1}{c^2}\left(\frac{ds'}{dt'}\right)^2\right]^{\frac{3}{2}}}$$

$$\frac{ds'}{dt'}K=\frac{ds'}{dt'}\frac{d\varphi}{ds}=v'K=v'\frac{d\varphi}{ds}=\frac{\frac{1}{c^2}\frac{ds'}{dt'}\frac{d^2s'}{dt'^2}}{\left[1+\frac{1}{c^2}\left(\frac{ds'}{dt'}\right)^2\right]^{\frac{3}{2}}}=\frac{\frac{1}{c^2}\vec{v}'\frac{d\vec{v}'}{dt'}}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}$$

$$\vec{v}'K=\vec{v}'\frac{d\vec{\varphi}}{ds}=\frac{\frac{1}{c^2}\vec{v}'\frac{d\vec{v}'}{dt'}}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} \qquad K=\frac{d\vec{\varphi}}{ds}=\frac{\frac{1}{c^2}\frac{d\vec{v}'}{dt'}}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}$$

$$dE_k=\frac{m_o v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_o v'\,dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=-\frac{k}{r^2}dr=\frac{k}{r^2}\hat{r}\,d\vec{r}' \qquad 21.56$$

$$\frac{dE_k}{dt'}=\vec{F}'\cdot\vec{v}'=\frac{m_o\frac{c^2}{c^2}\vec{v}'\frac{d\vec{v}'}{dt'}}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=\frac{k}{r^2}\hat{r}\frac{d\vec{r}'}{dt'}=\frac{k}{r^2}\hat{r}\vec{v}'$$

$$\frac{dE_k}{dt'}=\vec{F}'\cdot\vec{v}'=m_o c^2\vec{v}'\frac{d\vec{\varphi}}{ds}=\frac{k}{r^2}\hat{r}\vec{v}'$$

$$\vec{F}'=m_o c^2\frac{d\vec{\varphi}}{ds}=\frac{k}{r^2}\hat{r} \qquad K=\frac{d\vec{\varphi}}{ds}=\frac{k}{m_o c^2}\frac{1}{r^2}\hat{r}$$

§24 Variational Principle

$$E_k = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} = -\frac{k}{r} + constant \qquad 21.21$$

$$E_k = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2\sqrt{1-\frac{v^2}{c^2}} = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} = -\frac{k}{r} + constant$$

$$\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} - \left(-m_o c^2\sqrt{1-\frac{v^2}{c^2}} + \frac{k}{r}\right) = m_o c^2 \qquad p = \frac{d}{dv}\left(-m_o c^2\sqrt{1-\frac{v^2}{c^2}}\right) = \frac{m_o v}{\sqrt{1-\frac{v^2}{c^2}}}$$

$L = -m_o c^2\sqrt{1-\frac{v^2}{c^2}} + \frac{k}{r}$ Lagrangeana.

$\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} - L = m_o c^2$ What is the initial energy of the particle of mass mo.

$$pv - L = m_o c^2 \qquad L = pv - m_o c^2 = -m_o c^2\sqrt{1-\frac{v^2}{c^2}} + \frac{k}{r}$$

Variational Principle

$Ação = S = \int_{t_1}^{t_2} L[x(t), \dot{x}(t), t]\,dt \qquad \dot{x} = \frac{dx}{dt} = ux$ This is the velocity component in x axis.

$\delta S = \delta\int_{t_1}^{t_2} L(x, \dot{x}, t)\,dt = zero$ Variation of the action along the X axis.

Building the variable $x' = x + \varepsilon\eta$ in the range $t_1 \le t \le t_2$ we have seen this when $\varepsilon \to zero \Rightarrow x' = x$ and where $\varepsilon \ne zero$ we will have the conditions:

$$\frac{d\varepsilon}{dt} = zero \qquad \eta = \eta(t) \qquad \eta(t_1) = zero \qquad \eta(t_2) = zero \qquad \frac{d\eta}{d\varepsilon} = zero \qquad \dot{\eta} = \frac{d\eta}{dt}$$

$$x' = x + \varepsilon\eta \qquad \dot{x}' = \dot{x} + \varepsilon\dot{\eta} \qquad \frac{dx'}{d\varepsilon} = \eta \qquad \frac{d\dot{x}'}{d\varepsilon} = \dot{\eta} \qquad \frac{dx}{d\varepsilon} = zero \qquad \frac{d\dot{x}}{d\varepsilon} = zero$$

Then we have a new function $I(\varepsilon) = \int_{t_1}^{t_2} G(x + \varepsilon\eta, \dot{x} + \varepsilon\dot{\eta}, t)\,dt = \int_{t_1}^{t_2} F(x', \dot{x}', t)\,dt$ and where:

$$\varepsilon = zero \to x' = x \to \dot{x}' = \dot{x} \to F = L \Rightarrow \int_{t_1}^{t_2} F(x', \dot{x}', t)\,dt = \int_{t_1}^{t_2} L(x, \dot{x}, t)\,dt$$

$$\varepsilon \ne zero \to x' \ne x \to \dot{x}' \ne \dot{x} \to F \ne L \Rightarrow \int_{t_1}^{t_2} F(x', \dot{x}', t)\,dt \ne \int_{t_1}^{t_2} L(x, \dot{x}, t)\,dt$$

So we have $I(\varepsilon)=\int_{t_1}^{t_2} F[x'(\varepsilon),\dot{x}'(\varepsilon),t]dt$ that provides derived:

$$\frac{\partial I(\varepsilon)}{d\varepsilon}=\int_{t_1}^{t_2}\frac{\partial F(x',\dot{x}',t)}{\partial x'}\frac{dx'}{d\varepsilon}dt+\int_{t_1}^{t_2}\frac{\partial F(x',\dot{x}',t)}{\partial \dot{x}'}\frac{d\dot{x}'}{d\varepsilon}dt=\int_{t_1}^{t_2}\frac{\partial F}{\partial x'}\eta dt+\int_{t_1}^{t_2}\frac{\partial F}{\partial \dot{x}'}\dot{\eta}dt=zero$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}'}\eta\right)=\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}'}\right)\eta+\frac{\partial F}{\partial \dot{x}'}\frac{d\eta}{dt}\Rightarrow\frac{\partial F}{\partial \dot{x}'}\dot{\eta}=\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}'}\eta\right)-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}'}\right)\eta$$

$$\frac{\partial I(\varepsilon)}{d\varepsilon}=\int_{t_1}^{t_2}\frac{\partial F}{\partial x'}\eta dt+\int_{t_1}^{t_2}\frac{\partial F}{\partial \dot{x}'}\dot{\eta}dt=\int_{t_1}^{t_2}\frac{\partial F}{\partial x'}\eta dt+\int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}'}\eta\right)-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}'}\right)\eta\right]dt=zero$$

$$\frac{\partial I(\varepsilon)}{d\varepsilon}=\int_{t_1}^{t_2}\frac{\partial F}{\partial x'}\eta dt+\int_{t_1}^{t_2}d\left(\frac{\partial F}{\partial \dot{x}'}\eta\right)-\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}'}\right)\eta dt=zero$$

$$\int_{t_1}^{t_2}d\left(\frac{\partial F}{\partial \dot{x}'}\eta\right)=\frac{\partial F}{\partial \dot{x}'}\eta\Big|_{t_1}^{t_2}=\frac{\partial F}{\partial \dot{x}'}\eta(t_2)-\frac{\partial F}{\partial \dot{x}'}\eta(t_1)=zero$$

$$\frac{\partial I(\varepsilon)}{d\varepsilon}=\int_{t_1}^{t_2}\frac{\partial F}{\partial x'}\eta dt-\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}'}\right)\eta dt=\int_{t_1}^{t_2}\left[\frac{\partial F}{\partial x'}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}'}\right)\right]\eta dt=zero$$

$$\frac{\partial I(\varepsilon)}{d\varepsilon}=\int_{t_1}^{t_2}\left[\frac{\partial F}{\partial x'}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}'}\right)\right]\eta dt=zero\Rightarrow\eta\neq zero\rightarrow\frac{\partial F}{\partial x'}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}'}\right)=zero$$

$$\varepsilon=zero\rightarrow x'=x\rightarrow\dot{x}'=\dot{x}\rightarrow F=L\Rightarrow\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=zero$$

$\frac{\partial L}{\partial x}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$ This is the X axis component $\quad L=-m_o c^2\sqrt{1-\frac{v^2}{c^2}}+\frac{k}{r}$

$$\frac{\partial}{\partial x}\left(-m_o c^2\sqrt{1-\frac{v^2}{c^2}}+\frac{k}{r}\right)=\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(-m_o c^2\sqrt{1-\frac{v^2}{c^2}}+\frac{k}{r}\right)\right]$$

$\frac{\partial}{\partial x}\left(-m_o c^2\sqrt{1-\frac{v^2}{c^2}}\right)=zero \qquad \frac{\partial}{\partial \dot{x}}\left(\frac{k}{r}\right)=zero \qquad v=\sqrt{\frac{dx^2}{dt}+\frac{dy^2}{dt}+\frac{dz^2}{dt}}=\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$

$\frac{\partial}{\partial x}\left(\frac{k}{r}\right)=\frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}\left(-m_o c^2\sqrt{1-\frac{v^2}{c^2}}\right)\right]$ This is the X axis component

$\frac{\partial}{\partial x}\left(\frac{k}{r}\right)=k\frac{\partial}{\partial x}\left(r^{-1}\right)=k(-1)r^{-1-1}\frac{\partial r}{\partial x}=-k\frac{1}{r^2}\frac{x}{r}=-k\frac{x}{r^3} \qquad r^2=x^2+y^2+z^2$

$$\frac{\partial}{\partial \dot{x}}\left(-m_0 c^2\sqrt{1-\frac{v^2}{c^2}}\right)=-m_0 c^2\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}-1}\left(-\frac{2v}{c^2}\frac{dv}{d\dot{x}}\right)=\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d}{d\dot{x}}\left(\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\right)$$

$$\frac{\partial}{\partial \dot{x}}\left(-m_0 c^2\sqrt{1-\frac{v^2}{c^2}}\right)=\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\left[\frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)^{\frac{1}{2}-1}2\dot{x}\right]=\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right)=\frac{m_0\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{d}{dt}\left(\frac{m_0\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)}\left[\frac{d\dot{x}}{dt}\sqrt{1-\frac{v^2}{c^2}}-\dot{x}\frac{d}{dt}\left(\sqrt{1-\frac{v^2}{c^2}}\right)\right]=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)}\left[\frac{d\dot{x}}{dt}\sqrt{1-\frac{v^2}{c^2}}-\dot{x}\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}-1}\left(-\frac{2v}{c^2}\frac{dv}{dt}\right)\right]$$

$$\frac{d}{dt}\left(\frac{m_0\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)}\left[\frac{d\dot{x}}{dt}\sqrt{1-\frac{v^2}{c^2}}+\frac{\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{v}{c^2}\frac{dv}{dt}\right)\right]=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)}\left[\frac{\frac{d\dot{x}}{dt}\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{v}{c^2}\frac{dv}{dt}\right)\right]$$

$$\frac{d}{dt}\left(\frac{m_0\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)}\left[\frac{\frac{d\dot{x}}{dt}\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{v}{c^2}\frac{dv}{dt}\right)\right]=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\frac{d\dot{x}}{dt}+v\frac{dv}{dt}\frac{\dot{x}}{c^2}\right]$$

$$-k\frac{x}{r^3}\hat{i}=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\ddot{x}+v\frac{dv}{dt}\frac{\dot{x}}{c^2}\right]\hat{i}$$ X axis

$$-k\frac{y}{r^3}\hat{j}=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\ddot{y}+v\frac{dv}{dt}\frac{\dot{y}}{c^2}\right]\hat{j}$$ Y axis

$$-k\frac{z}{r^3}\hat{k}=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\ddot{z}+v\frac{dv}{dt}\frac{\dot{z}}{c^2}\right]\hat{k}$$ Z axis

$$-k\frac{x}{r^3}\hat{i}-k\frac{y}{r^3}\hat{j}-k\frac{z}{r^3}\hat{k}=\frac{-k}{r^3}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)=\frac{-k}{r^3}\vec{r}=\frac{-k}{r^2}\hat{r}$$

$$\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\ddot{x}+v\frac{dv}{dt}\frac{\dot{x}}{c^2}\right]\hat{i}+\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\ddot{y}+v\frac{dv}{dt}\frac{\dot{y}}{c^2}\right]\hat{j}+\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\ddot{z}+v\frac{dv}{dt}\frac{\dot{z}}{c^2}\right]\hat{k}=\frac{-k}{r^2}\hat{r}$$

$$\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left\{\left[\left(1-\frac{v^2}{c^2}\right)\ddot{x}+v\frac{dv}{dt}\frac{\dot{x}}{c^2}\right]\hat{i}+\left[\left(1-\frac{v^2}{c^2}\right)\ddot{y}+v\frac{dv}{dt}\frac{\dot{y}}{c^2}\right]\hat{j}+\left[\left(1-\frac{v^2}{c^2}\right)\ddot{z}+v\frac{dv}{dt}\frac{\dot{z}}{c^2}\right]\hat{k}\right\}=\frac{-k}{r^2}\hat{r}$$

$$\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\ddot{x}\hat{i}+v\frac{dv}{dt}\frac{\dot{x}}{c^2}\hat{i}+\left(1-\frac{v^2}{c^2}\right)\ddot{y}\hat{j}+v\frac{dv}{dt}\frac{\dot{y}}{c^2}\hat{j}+\left(1-\frac{v^2}{c^2}\right)\ddot{z}\hat{k}+v\frac{dv}{dt}\frac{\dot{x}}{c^2}\hat{k}\right]=\frac{-k}{r^2}\hat{r}$$

$$\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\left(\ddot{x}\hat{i}+\ddot{y}\hat{j}+\ddot{z}\hat{k}\right)+\frac{v}{c^2}\frac{dv}{dt}\left(\dot{x}\hat{i}++\dot{y}\hat{j}+\dot{z}\hat{k}\right)\right]=\frac{-k}{r^2}\hat{r}$$

$$\bar{a}=\ddot{x}\hat{i}+\ddot{y}\hat{j}+\ddot{z}\hat{k}=\frac{d}{dt}\left(\dot{x}\hat{i}+\dot{y}\hat{j}+\dot{z}\hat{k}\right)=\frac{d\bar{v}}{dt} \qquad \bar{v}=\dot{x}\hat{i}++\dot{y}\hat{j}+\dot{z}\hat{k}$$

$$\bar{F}=\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\frac{d\bar{v}}{dt}+v\frac{dv}{dt}\frac{\bar{v}}{c^2}\right]=\frac{-k}{r^2}\hat{r}$$ = 21.16

$$\bar{F}=\frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v^2}{c^2}\right)\frac{d\bar{v}}{dt}+v\frac{dv}{dt}\frac{\bar{v}}{c^2}\right]=\frac{-k}{r^2}\hat{r}$$ = 21.19

§24 Variational Principle Continuation

$$E_k=m_oc^2\sqrt{1+\frac{v'^2}{c^2}}=\frac{m_oc^2}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{k}{r}+\text{constante}$$ 21.21

$$E_k=m_oc^2\sqrt{1+\frac{v'^2}{c^2}}=\frac{m_ov^2}{\sqrt{1-\frac{v^2}{c^2}}}+m_oc^2\sqrt{1-\frac{v^2}{c^2}}=\frac{m_oc^2}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{k}{r}+\text{constante}$$

$$E_k-\frac{k}{r}=m_oc^2\sqrt{1+\frac{v'^2}{c^2}}-\frac{k}{r}=\frac{m_ov^2}{\sqrt{1-\frac{v^2}{c^2}}}+m_oc^2\sqrt{1-\frac{v^2}{c^2}}-\frac{k}{r}=\frac{m_oc^2}{\sqrt{1-\frac{v^2}{c^2}}}-\frac{k}{r}=\frac{k}{r}-\frac{k}{r}+\text{constante}$$

$$E_k-\frac{k}{r}=m_oc^2\sqrt{1+\frac{v'^2}{c^2}}-\frac{k}{r}=\frac{m_ov^2}{\sqrt{1-\frac{v^2}{c^2}}}-\left(-m_oc^2\sqrt{1-\frac{v^2}{c^2}}+\frac{k}{r}\right)=m_oc^2=\text{constante}$$

$$T=m_oc^2\sqrt{1+\frac{v'^2}{c^2}} \qquad T=-m_oc^2\sqrt{1-\frac{v^2}{c^2}} \qquad E_p=-\frac{k}{r} \qquad pv=\frac{m_ov^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$pv=\frac{m_ov}{\sqrt{1-\frac{v^2}{c^2}}}v=v'\frac{m_ov'}{\sqrt{1+\frac{v'^2}{c^2}}}=v'p'. \qquad p=p'\sqrt{1+\frac{v'^2}{c^2}} \qquad p'=p\sqrt{1-\frac{v^2}{c^2}}$$

$$E_R=E_k+E_p=T'+E_p=pv-\left(T-E_p\right)$$

$$E_k=T' \qquad E_k=pv-T \qquad T'=pv-T \qquad T=p'v'-T'$$

$L' = T + E_p$ $\qquad L = T - E_p$

$E_R = E_k + E_p = L' = pv - L$

$L' = pv - L \qquad L = p'v' - L' \qquad L + L' = pv = p'v'$

$$p' = \frac{dT'}{dv'} = \frac{d}{dv'}\left(m_o c^2\sqrt{1+\frac{v'^2}{c^2}}\right) = \frac{m_o v'}{\sqrt{1+\frac{v'^2}{c^2}}} = m_o v \qquad p = \frac{dT}{dv} = \frac{d}{dv}\left(-m_o c^2\sqrt{1-\frac{v^2}{c^2}}\right) = \frac{m_o v}{\sqrt{1-\frac{v^2}{c^2}}} = m_o v'$$

$$d\vec{r}' = dx'\hat{i} + dy'\hat{j} + dz'\hat{k} = -dx\hat{i} - dy\hat{j} - dz\hat{k} = -d\vec{r}$$ 21.08

$$\vec{v}' = \frac{d\vec{r}'}{dt'} = \frac{dx'}{dt'}\hat{i} + \frac{dy'}{dt'}\hat{j} + \frac{dz'}{dt'}\hat{k} = \frac{-1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) = \frac{-1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{r}}{dt} = \frac{-\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\dot{x}' = v'_{x'} = \frac{dx'}{dt'} = \frac{-1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{dx}{dt} = \frac{-v_x}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{-\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$p'_x = \frac{dT'}{d\dot{x}'} = \frac{d}{d\dot{x}'}\left(m_o c^2\sqrt{1+\frac{v'^2}{c^2}}\right) = \frac{m_o \dot{x}'}{\sqrt{1+\frac{v'^2}{c^2}}} = -m_o\dot{x} \qquad p_x = \frac{dT}{d\dot{x}} = \frac{d}{d\dot{x}}\left(-m_o c^2\sqrt{1-\frac{v^2}{c^2}}\right) = \frac{m_o\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}} = -m_o\dot{x}'$$

$$\vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k} = -x\hat{i} - y\hat{j} - z\hat{k} = -\vec{r}$$ 21.07

$x' = -x \qquad y' = -y \qquad z' = -z$

$$\frac{\partial x'}{\partial x} = -1 \qquad \frac{\partial y'}{\partial y} = -1 \qquad \frac{\partial z'}{\partial z} = -1$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \text{zero}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial x'}{\partial x}\frac{\partial L}{\partial x'} - \frac{dt'}{dt}\frac{d}{dt'}\left(\frac{\partial L}{\partial \dot{x}}\right) = \text{zero} \qquad L = p'v' - L' \qquad \frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = p_x = -m_o\dot{x}'$$

$$\frac{\partial x'}{\partial x}\frac{\partial L}{\partial x'} - \frac{dt'}{dt}\frac{d}{dt'}\left(\frac{\partial L}{\partial \dot{x}}\right) = -\frac{\partial}{\partial x'}(p'v' - L') - \frac{1}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{d}{dt'}(-m_o\dot{x}') = \text{zero}$$

$$-\frac{\partial}{\partial x'}(p'v' - L') + \frac{m_o}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{d\dot{x}'}{dt'} = -v'\frac{\partial p'}{\partial x'} - p'\frac{\partial v'}{\partial x'} + \frac{\partial L'}{\partial x'} + \frac{m_o}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{d\dot{x}'}{dt'} = \text{zero}$$

$$\frac{\partial p'}{\partial x'} = \text{zero} \qquad \frac{\partial v'}{\partial x'} = \text{zero} \qquad L' = m_o c^2\sqrt{1+\frac{v'^2}{c^2}} - \frac{k}{r}$$

$$\frac{\partial L'}{\partial x'} + \frac{m_o}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{d\dot{x}'}{dt'} = \text{zero} \qquad r^2 = \vec{r}'.\vec{r}' = (-\vec{r}).(-\vec{r}) = x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$$

$$\frac{\partial L'}{\partial x'}=\frac{\partial}{\partial x'}\left(m_o c^2\sqrt{1+\frac{v'^2}{c^2}}-\frac{k}{r}\right)=\frac{\partial}{\partial x'}\left(-\frac{k}{r}\right)=-k\frac{\partial}{\partial x'}\left(r^{-1}\right)=-k(-1)r^{-1-1}\frac{\partial r}{\partial x'}=k\frac{1}{r^2}\frac{x'}{r}=k\frac{x'}{r^3}$$

$$\frac{\partial L'}{\partial x'}+\frac{m_o}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{d\dot{x}'}{dt'}=k\frac{x'}{r^3}+\frac{m_o\ddot{x}'}{\sqrt{1+\frac{v'^2}{c^2}}}=zero$$

$$\frac{m_o\ddot{x}'}{\sqrt{1+\frac{v'^2}{c^2}}}=-k\frac{x'}{r^3} \qquad -k\frac{x'}{r^3}\hat{i}-k\frac{y'}{r^3}\hat{j}-k\frac{z'}{r^3}\hat{k}=\frac{-k}{r^3}\left(x'\hat{i}+y'\hat{j}+z'\hat{k}\right)=\frac{-k}{r^3}\vec{r}'=\frac{-k}{r^2}\hat{r}'$$

$$\frac{m_o\ddot{x}'}{\sqrt{1+\frac{v'^2}{c^2}}}\hat{i}+\frac{m_o\ddot{y}'}{\sqrt{1+\frac{v'^2}{c^2}}}\hat{j}+\frac{m_o\ddot{z}'}{\sqrt{1+\frac{v'^2}{c^2}}}\hat{k}=\frac{m_o\vec{a}'}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{-k}{r^2}\hat{r}'$$

$$\frac{m_o\vec{a}'}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{-k}{r^2}\hat{r}'=\frac{k}{r^2}\hat{r} \qquad -\hat{r}'=\hat{r} \qquad \frac{-m_o\vec{a}'}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{-k}{r^2}\hat{r}$$ = 21.19

§25 Logarithmic spiral

$$H\frac{d^2w}{d\phi^2}+Hw+3A\frac{d^2w}{d\phi^2}w+3Aw^2-B=zero \qquad r=e^{a\phi}$$ 21.37

$$w=\frac{1}{r}=\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right] \qquad \frac{dw}{d\phi}=\frac{-Q\sin(\phi Q)}{D} \qquad \frac{d^2w}{d\phi^2}=\frac{-Q^2\cos(\phi Q)}{D}$$ 21.38

$$w=\frac{1}{r}=\frac{1}{e^{a\phi}}=e^{-a\phi} \qquad \frac{dw}{d\phi}=-ae^{-a\phi} \qquad \frac{d^2w}{d\phi^2}=a^2e^{-a\phi}$$

$$Ha^2e^{-a\phi}+He^{-a\phi}+3Aa^2e^{-a\phi}e^{-a\phi}+3A\left(e^{-a\phi}\right)^2-B=zero$$

$$Ha^2e^{-a\phi}+He^{-a\phi}+3Aa^2e^{-2a\phi}+3Ae^{-2a\phi}-B=zero$$

$$\left(1+a^2\right)He^{-a\phi}+\left(1+a^2\right)3Ae^{-2a\phi}-B=zero$$

$$\left(1+a^2\right)3Ae^{-2a\phi}+\left(1+a^2\right)He^{-a\phi}-B=zero$$

$$\left(1+a^2\right)3Aw^2+\left(1+a^2\right)Hw-B=zero$$

$$3Aw^2+Hw-\frac{B}{\left(1+a^2\right)}=zero$$

$$w=e^{-a\phi}=\frac{1}{r}=\frac{-H\pm\sqrt{H^2+4.3A\frac{B}{\left(1+a^2\right)}}}{2.3A}=\frac{-H}{6A}\pm\frac{1}{6A}\sqrt{H^2+\frac{12AB}{\left(1+a^2\right)}}$$

$$3A\left[\frac{-H}{6A}\pm\frac{1}{6A}\sqrt{H^2+\frac{12AB}{\left(1+a^2\right)}}\right]^2+H\left[\frac{-H}{6A}\pm\frac{1}{6A}\sqrt{H^2+\frac{12AB}{\left(1+a^2\right)}}\right]-\frac{B}{\left(1+a^2\right)}=zero$$

$$3A\left[\left(\frac{-H}{6A}\right)^2 \pm 2\left(\frac{-H}{6A}\right)\frac{1}{6A}\sqrt{H^2+\frac{12AB}{(1+a^2)}}+\left(\frac{1}{6A}\sqrt{H^2+\frac{12AB}{(1+a^2)}}\right)^2\right]-$$

$$\frac{-H^2}{6A}\pm\frac{H}{6A}\sqrt{H^2+\frac{12AB}{(1+a^2)}}-\frac{B}{(1+a^2)}=zero$$

$$3A\left[\left(\frac{-H}{6A}\right)^2 \pm 2\left(\frac{-H}{6A}\right)\frac{1}{6A}\sqrt{H^2+\frac{12AB}{(1+a^2)}}+\frac{1}{36A^2}\left(H^2+\frac{12AB}{(1+a^2)}\right)\right]-$$

$$\frac{-H^2}{6A}\pm\frac{H}{6A}\sqrt{H^2+\frac{12AB}{(1+a^2)}}-\frac{B}{(1+a^2)}=zero$$

$$3A\left[\frac{H^2}{36A^2}\pm\frac{-H}{18A^2}\sqrt{H^2+\frac{12AB}{(1+a^2)}}+\frac{1}{36A^2}\left(H^2+\frac{12AB}{(1+a^2)}\right)\right]-$$

$$\frac{-H^2}{6A}\pm\frac{H}{6A}\sqrt{H^2+\frac{12AB}{(1+a^2)}}-\frac{B}{(1+a^2)}=zero$$

$$\frac{H^2}{12A}\pm\frac{-H}{6A}\sqrt{H^2+\frac{12AB}{(1+a^2)}}+\frac{1}{12A}\left(H^2+\frac{12AB}{(1+a^2)}\right)-$$

$$\frac{-H^2}{6A}\pm\frac{H}{6A}\sqrt{H^2+\frac{12AB}{(1+a^2)}}-\frac{B}{(1+a^2)}=zero$$

$$\frac{H^2}{12A}+\frac{1}{12A}\left(H^2+\frac{12AB}{(1+a^2)}\right)-\frac{H^2}{6A}-\frac{B}{(1+a^2)}=zero$$

$$\frac{H^2}{12A}+\frac{H^2}{12A}+\frac{B}{(1+a^2)}-\frac{H^2}{6A}-\frac{B}{(1+a^2)}=zero$$

§25 Logarithmic Spiral (Continuation)

$$-\left(H+A\frac{1}{r}\right)^3\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{GM_o}{L'^2} \qquad 21.71$$

$$\left(H+A\frac{1}{r}\right)^3\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{-GM_o}{L'^2}$$

$$\left(H+A\frac{1}{r}\right)^3\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=-B \qquad H=\frac{E_R}{m_o c^2} \qquad A=\frac{GM_o}{c^2} \qquad B=\frac{GM_o}{L'^2}$$

$$\left(H+A\frac{1}{r}\right)^3\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)+B=zero$$

$$\left(H^3+3H^2A\frac{1}{r}+3HA^2\frac{1}{r^2}+A^3\frac{1}{r^3}\right)\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)+B=zero$$

$$H^3+3H^2A\frac{1}{r}+3HA^2\frac{1}{r^2}+A^3\frac{1}{r^3}\cong H^3+3H^2A\frac{1}{r} \qquad 3HA^2\frac{1}{r^2}+A^3\frac{1}{r^3}\cong zero$$

$$\left(H^3+3AH^2\frac{1}{r}\right)\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)+B=zero$$

$$\left(H^3+3AH^2w\right)\left(\frac{d^2w}{d\phi^2}+w\right)+B=zero$$

$$H^3\frac{d^2w}{d\phi^2}+H^3w+3AH^2\frac{d^2w}{d\phi^2}w+3AH^2w^2+B=zero$$

$$w=\frac{1}{r}=\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)] \qquad \frac{dw}{d\phi}=\frac{-Q\,sen(\phi Q)}{D} \qquad \frac{d^2w}{d\phi^2}=\frac{-Q^2\cos(\phi Q)}{D} \qquad 21.38$$

$$H^3\left[\frac{-Q^2\cos(\phi Q)}{D}\right]+H^3\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]+3AH^2\left[\frac{-Q^2\cos(\phi Q)}{D}\right]\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]+$$
$$+3AH^2\left\{\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]\right\}^2+B=zero$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D}+H^3\frac{1}{\varepsilon D}+H^3\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q)+3AH^2\left[\frac{-Q^2\cos(\phi Q)}{D}\right]\frac{1}{\varepsilon D}+3AH^2\left[\frac{-Q^2\cos(\phi Q)}{D}\right]\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q)+$$
$$+3AH^2\left\{\frac{1}{\varepsilon^2D^2}\left[1+2\varepsilon\cos(\phi Q)+\varepsilon^2\cos^2(\phi Q)\right]\right\}+B=zero$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D}+\frac{H^3}{\varepsilon D}+H^3\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-3AH^2Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{\varepsilon^2D^2}\left[1+2\varepsilon\cos(\phi Q)+\varepsilon^2\cos^2(\phi Q)\right]+B=zero$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D}+\frac{H^3}{\varepsilon D}+H^3\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-3AH^2Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{\varepsilon^2D^2}+\frac{3AH^2}{\varepsilon^2D^2}2\varepsilon\cos(\phi Q)+\frac{3AH^2}{\varepsilon^2D^2}\varepsilon^2\cos^2(\phi Q)+B=zero$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D}+\frac{H^3}{\varepsilon D}+H^3\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-3AH^2Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{\varepsilon^2D^2}+\frac{6AH^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+3AH^2\frac{\cos^2(\phi Q)}{D^2}+B=zero$$

$$\frac{-H^3Q^2}{3AH^2}\frac{\cos(\phi Q)}{D}+\frac{H^3}{3AH^2\varepsilon D}+\frac{H^3}{3AH^2}\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{3AH^2\varepsilon D}\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{3AH^2}\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{3AH^2\varepsilon^2D^2}+\frac{6AH^2}{3AH^2\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{3AH^2}{3AH^2}\frac{\cos^2(\phi Q)}{D^2}+\frac{B}{3AH^2}=zero$$

$$\frac{-HQ^2}{3A}\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{H}{3A}\frac{\cos(\phi Q)}{D}-\frac{Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{1}{\varepsilon^2D^2}+\frac{2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{\cos^2(\phi Q)}{D^2}+\frac{B}{3AH^2}=zero$$

$$\frac{\cos^2(\phi Q)}{D^2}-Q^2\frac{\cos^2(\phi Q)}{D^2}-\frac{HQ^2}{3A}\frac{\cos(\phi Q)}{D}+\frac{H}{3A}\frac{\cos(\phi Q)}{D}-\frac{Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+$$

$$+\frac{2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{B}{3AH^2}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-\frac{HQ^2}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{B}{3AH^2}=\text{zero}$$

$$H=\frac{E_R}{m_o c^2}=\frac{-m_o c^2}{m_o c^2}=-1 \qquad Q^2=1-\frac{6A}{\varepsilon D}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-\frac{(-1)Q^2}{3A}+\frac{(-1)}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{(-1)}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{B}{3A(-1)^2}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}-\frac{1}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{B}{3A}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}-\frac{1}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{\varepsilon DB}{3A\varepsilon D}=\text{zero}$$

$$\varepsilon DB=\frac{\varepsilon DGM_o}{L'^2}=1$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}-\frac{1}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{1}{3A\varepsilon D}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=\text{zero}$$

$$Q^2=1-\frac{6A}{\varepsilon D}$$

$$\left[1-\left(1-\frac{6A}{\varepsilon D}\right)\right]\frac{\cos^2(\phi Q)}{D^2}+\left[\frac{1}{3A}\left(1-\frac{6A}{\varepsilon D}\right)-\frac{1}{3A}-\frac{1}{\varepsilon D}\left(1-\frac{6A}{\varepsilon D}\right)+\frac{2}{\varepsilon D}\right]\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=\text{zero}$$

$$\left(1-1+\frac{6A}{\varepsilon D}\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{1}{3A}-\frac{1}{3A}\frac{6A}{\varepsilon D}-\frac{1}{3A}-\frac{1}{\varepsilon D}+\frac{1}{\varepsilon D}\frac{6A}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=\text{zero}$$

$$\left(\frac{6A}{\varepsilon D}\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-\frac{2}{\varepsilon D}+\frac{6A}{\varepsilon^2 D^2}+\frac{1}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=\text{zero}$$

$$(6A)\left(\frac{1}{\varepsilon D}\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-\frac{1}{\varepsilon D}+\frac{6A}{\varepsilon^2 D^2}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=\text{zero}$$

$$(6A)\left(\frac{1}{\varepsilon D}\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-1+\frac{6A}{\varepsilon D}\right)\left(\frac{1}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\left(\frac{1}{\varepsilon D}\right)\left(\frac{1}{\varepsilon D}\right)=\text{zero}$$

$$(6A)\frac{\cos^2(\phi Q)}{D^2}+\left(-1+\frac{6A}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon D}=\text{zero}$$

$$\frac{\cos(\phi Q)}{D}=\frac{-\left(-1+\frac{6A}{\varepsilon D}\right)\pm\sqrt{\left(-1+\frac{6A}{\varepsilon D}\right)^2-4.6A.\frac{1}{\varepsilon D}}}{2.6A}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\left(1-\frac{6A}{\varepsilon D}\right)\pm\sqrt{(-1)^2+2(-1)\frac{6A}{\varepsilon D}+\left(\frac{6A}{\varepsilon D}\right)^2-\frac{24A}{\varepsilon D}}}{12A}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\left(1-\frac{6A}{\varepsilon D}\right)\pm\sqrt{1-\frac{12A}{\varepsilon D}+\left(\frac{6A}{\varepsilon D}\right)^2-\frac{24A}{\varepsilon D}}}{12A}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\left(1-\frac{6A}{\varepsilon D}\right)\pm\sqrt{1-\frac{36A}{\varepsilon D}+\frac{36A^2}{\varepsilon^2 D^2}}}{12A}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\left(1-\frac{6A}{\varepsilon D}\right)\pm\sqrt{1-\frac{36A}{\varepsilon D}+\frac{36A^2}{\varepsilon^2 D^2}}}{12A}$$

$$\sqrt{1-\frac{36A}{\varepsilon D}+\frac{36A^2}{\varepsilon^2 D^2}}\simeq\sqrt{1-\frac{36A}{\varepsilon D}} \qquad \frac{36A^2}{\varepsilon^2 D^2}\simeq\text{zero} \qquad A=\frac{GM_o}{c^2}$$

$$\frac{36A^2}{\varepsilon^2 D^2}=\frac{36}{\varepsilon^2 D^2}\left(\frac{GM_o}{c^2}\right)^2=\frac{36}{(55442955600)^2}\left[\frac{6{,}67.10^{-11}\,1{,}989.10^{30}}{\left(2{,}99792458.10^{8}\right)^2}\right]^2=2{,}55.10^{-14}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\left(1-\frac{6A}{\varepsilon D}\right)\pm\sqrt{1-\frac{36A}{\varepsilon D}}}{12A}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\left(1-\frac{6A}{\varepsilon D}\right)\pm\sqrt{1-\frac{36A}{\varepsilon D}}}{12A}=\frac{1-\frac{6A}{\varepsilon D}\pm\left(1-\frac{1}{2}\frac{36A}{\varepsilon D}\right)}{12A}=\frac{1-\frac{6A}{\varepsilon D}\pm\left(1-\frac{18A}{\varepsilon D}\right)}{12A}$$

$$\frac{\cos(\phi Q)}{D}=\frac{1-\frac{6A}{\varepsilon D}-\left(1-\frac{18A}{\varepsilon D}\right)}{12A}=\frac{1-\frac{6A}{\varepsilon D}-1+\frac{18A}{\varepsilon D}}{12A}=\frac{\frac{12A}{\varepsilon D}}{12A}=\frac{1}{\varepsilon D}$$

$$-\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon D}=\text{zero}$$

$$\text{zero}<r(\phi Q)<\infty\rightarrow M_o\neq\text{zero}\rightarrow Q=\sqrt{1-\frac{6A}{\varepsilon D}}\rightarrow-\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon D}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=zero$$

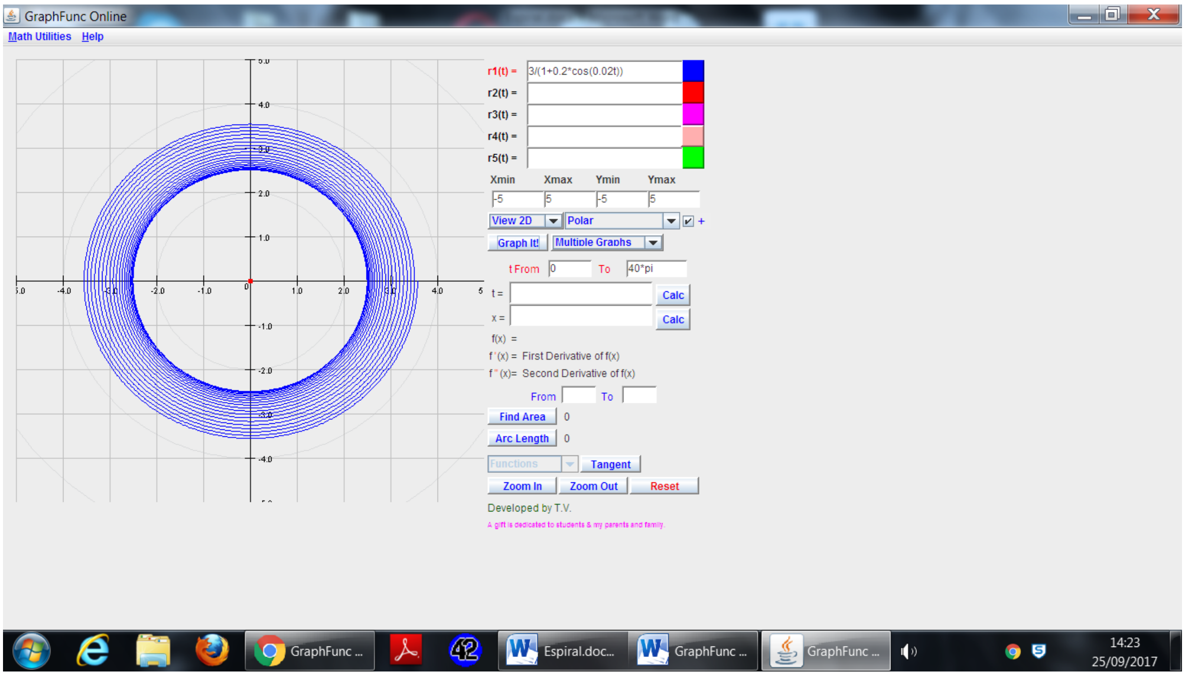
$$r=\infty\to M_0=zero\to Q=1 \qquad Q=\sqrt{1-\frac{6A}{\varepsilon D}}=\sqrt{1-\frac{6}{\varepsilon D}\left(\frac{GM_0}{c^2}\right)}=\sqrt{1-\frac{6}{\varepsilon D}\left(\frac{G(zero)}{c^2}\right)}=1$$

$$(1-1)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{1}{3A}-\frac{1}{3A}-\frac{1}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=zero$$

$$\left(\frac{1}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=zero \qquad \frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon D}=zero$$

$$r=\infty\to M_0=zero\to Q=1\to w=\frac{1}{r=\infty}=\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]=\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon D}=zero$$

The presence of Q in the formula $r=r(\phi Q)=\frac{\varepsilon D}{1+\varepsilon\cos(\phi Q)}$, allows it to also describe a spiral.



§25 Logarithmic Spiral Continuation II

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=zero$$

$$zero<r(\phi Q)<\infty\to M_0\neq zero\to Q=\frac{\sqrt{1-\frac{12A}{\varepsilon D}}}{\sqrt{1-\frac{6A}{\varepsilon D}}}$$

$$\left[1-\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)\right]\frac{\cos^2(\phi Q)}{D^2}+\left[\frac{1}{3A}\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)-\frac{1}{3A}-\frac{1}{\varepsilon D}\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)+\frac{2}{\varepsilon D}\right]\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=zero$$

$$\left[1-\frac{6A}{\varepsilon D}-\left(1-\frac{12A}{\varepsilon D}\right)\right]\frac{\cos^2(\phi Q)}{D^2}+\left[\frac{1}{3A}\left(1-\frac{12A}{\varepsilon D}\right)-\frac{1}{3A}\left(1-\frac{6A}{\varepsilon D}\right)-\frac{1}{\varepsilon D}\left(1-\frac{12A}{\varepsilon D}\right)+\frac{2}{\varepsilon D}\left(1-\frac{6A}{\varepsilon D}\right)\right]\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}\left(1-\frac{6A}{\varepsilon D}\right)=\text{zero}$$

$$\left(1-\frac{6A}{\varepsilon D}-1+\frac{12A}{\varepsilon D}\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{1}{3A}-\frac{1}{3A}\frac{12A}{\varepsilon D}-\frac{1}{3A}+\frac{1}{3A}\frac{6A}{\varepsilon D}-\frac{1}{\varepsilon D}+\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}+\frac{2}{\varepsilon D}-\frac{2}{\varepsilon D}\frac{6A}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}-\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}=\text{zero}$$

$$\left(\frac{6A}{\varepsilon D}\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-\frac{1}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}-\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}=\text{zero}$$

$$\frac{\cos(\phi Q)}{D}=\frac{-\left(-\frac{1}{\varepsilon D}\right)\pm\sqrt{\left(-\frac{1}{\varepsilon D}\right)^2-4\frac{6A}{\varepsilon D}\left(\frac{1}{\varepsilon^2 D^2}-\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}\right)}}{2\frac{6A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\sqrt{\frac{1}{\varepsilon^2 D^2}-\frac{24A}{\varepsilon D}\left(\frac{1}{\varepsilon^2 D^2}-\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}\right)}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\sqrt{\frac{1}{\varepsilon^2 D^2}-\frac{24A}{\varepsilon D}\frac{1}{\varepsilon^2 D^2}+\frac{24A}{\varepsilon D}\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\sqrt{1-\frac{24A}{\varepsilon D}+\frac{24A}{\varepsilon D}\frac{6A}{\varepsilon D}}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\sqrt{1-2\frac{12A}{\varepsilon D}+\frac{144A^2}{\varepsilon^2 D^2}}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\sqrt{\left(1-\frac{12A}{\varepsilon D}\right)^2}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\left(1-\frac{12A}{\varepsilon D}\right)}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\left(\frac{1}{\varepsilon D}-\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}\right)}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}-\left(\frac{1}{\varepsilon D}-\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}\right)}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}-\frac{1}{\varepsilon D}+\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{1}{\varepsilon D}$$

$$-\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon D}=zero$$

$$zero<r(\phi Q)<\infty\rightarrow M_o\neq zero\rightarrow Q=\frac{\sqrt{1-\frac{12A}{\varepsilon D}}}{\sqrt{1-\frac{6A}{\varepsilon D}}}\rightarrow -\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon D}=zero$$

$$Q^2=\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\approx 1-\frac{6A}{\varepsilon D}\qquad Q^2=1-\frac{6A}{\varepsilon D}\qquad A=\frac{GM_o}{c^2}$$

$$\varepsilon D=a(1-\varepsilon^2)=57.909.227.000,00\left[1-(0,20563593)^2\right]=55.460.469.568,40$$

$$A=\frac{GM_o}{c^2}=\frac{6,67408311\cdot 10^{-11}\cdot 1,9891\cdot 10^{30}}{\left(2,99792458\cdot 10^{8}\right)^2}=1.477,08953542$$

$$Q=\sqrt{\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}}=0,9999999201\qquad Q=\sqrt{1-\frac{6A}{\varepsilon D}}=0,9999999201$$

$1,276.789.102.53^{-14}$

$$\phi Q=1.296.000,00\Rightarrow\phi=\frac{1.296.000,00}{Q}$$

$Q<1$ Advance $\qquad Q>1$ Regression

$$\Delta\phi=\left(\frac{1}{Q}-1\right)1.296.000,00$$

$\Delta\phi>zero$ Advance $\qquad \Delta\phi<zero$ Regression

$$\Delta\phi=\left[\frac{1}{\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)^{\frac{1}{2}}}-1\right]1.296.000,00=0,103549893544$$

$$\Delta\phi = \left[\frac{1}{\left(1-\frac{6A}{\varepsilon D}\right)^{\frac{1}{2}}} - 1 \right] 1.296.000,00 = 0,103549876997''$$

$$N = 100\frac{PT}{PM} = 100\frac{365,256363004}{87,969} = 415,210316139$$

$$\sum \Delta\phi = \Delta\phi N = 0,103549893544 \text{ x } 415,210316139 = 42,994984034 7''$$

$$\sum \Delta\phi = \Delta\phi N = 0,103549876997 \text{ x } 415,210316139 = 42,994977164 2''$$

By definition $\varepsilon > zero$

$$zero < r(\phi Q) < \infty \rightarrow M_o \neq zero \rightarrow Q = \sqrt{\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}} \qquad r = \infty \rightarrow M_o = zero \rightarrow Q = 1$$

$$-\frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = zero \rightarrow \varepsilon = \frac{1}{\cos(\phi Q)} \qquad \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = zero \rightarrow \varepsilon = \frac{-1}{\cos(\phi Q)}$$

Se If $Q = 1$

$$\left[\varepsilon = \frac{1}{\cos(\phi - \pi)}\right] = \left[\varepsilon = \frac{-1}{\cos(\phi)}\right]$$

Energy Newtonian (E_N)

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2} + \left(x - \frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D} + \frac{Q^2}{D^2} + \frac{1}{\varepsilon^2 D^2} - \frac{x}{\varepsilon D} - y = zero$$

$$r = \infty \rightarrow Q = 1 \rightarrow w = \frac{1}{r = \infty} = \frac{1}{\varepsilon D}[1 + \varepsilon\cos(\phi Q)] = \frac{\cos(\phi Q)}{D} + \frac{1}{\varepsilon D} = zero$$

$$\left(1-Q^2\right)\left(-\frac{1}{\varepsilon D}\right)\left(-\frac{1}{\varepsilon D}\right) + \left(x - \frac{2}{\varepsilon D}\right)\left(-\frac{1}{\varepsilon D}\right) + \frac{Q^2}{D^2} + \frac{1}{\varepsilon^2 D^2} - \frac{x}{\varepsilon D} - y = zero$$

$$\left(1-Q^2\right)\left(\frac{1}{\varepsilon^2 D^2}\right) - \frac{x}{\varepsilon D} + \frac{2}{\varepsilon^2 D^2} + \frac{Q^2}{D^2} + \frac{1}{\varepsilon^2 D^2} - \frac{x}{\varepsilon D} - y = zero$$

$$\frac{1}{\varepsilon^2 D^2} - \frac{Q^2}{\varepsilon^2 D^2} - \frac{x}{\varepsilon D} + \frac{2}{\varepsilon^2 D^2} + \frac{Q^2}{D^2} + \frac{1}{\varepsilon^2 D^2} - \frac{x}{\varepsilon D} - y = zero$$

$$-\frac{Q^2}{\varepsilon^2 D^2} + \frac{Q^2}{D^2} + \frac{4}{\varepsilon^2 D^2} - \frac{2x}{\varepsilon D} - y = zero \qquad Q^2 = 1$$

$$-\frac{1}{\varepsilon^2 D^2} + \frac{1}{D^2} + \frac{4}{\varepsilon^2 D^2} - \frac{2x}{\varepsilon D} - y = zero$$

$$-\frac{\varepsilon^2 D^2}{\varepsilon^2 D^2}+\frac{\varepsilon^2 D^2}{D^2}+\frac{4\varepsilon^2 D^2}{\varepsilon^2 D^2}-\frac{2x\varepsilon^2 D^2}{\varepsilon D}-\varepsilon^2 D^2 y=zero$$

$$-1+\varepsilon^2+4-2x\varepsilon D-\varepsilon^2 D^2 y=zero$$

$$x=\frac{2}{\varepsilon D} \qquad y=\frac{2E_N}{m_o L^2} \qquad L^2=\varepsilon DGM \qquad \frac{1}{a}=\frac{-1}{\varepsilon D}\left(\varepsilon^2-1\right)$$

$$-1+\varepsilon^2+4-2\frac{2}{\varepsilon D}\varepsilon D-\varepsilon^2 D^2 y=zero \qquad -1+\varepsilon^2-\varepsilon^2 D^2 y=zero$$

$$-1+\varepsilon^2-\varepsilon^2 D^2\frac{2E_N}{m_o L^2}=zero \qquad -1+\varepsilon^2-\varepsilon^2 D^2\frac{2E_N}{m_o \varepsilon DGM_o}=zero$$

$$-1+\varepsilon^2-\varepsilon D\frac{2E_N}{GM_o m_o}=zero \qquad \frac{1}{\varepsilon D}\left(\varepsilon^2-1\right)=\frac{2E_N}{k}$$

$$E_N=\frac{-k}{2a}$$

§26 Advancement of the Periélio of Mercury of 42,99 "

Supposing $ux=v$

(2.3) $$u'x'=\frac{ux-v}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}}=\frac{v-v}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vv}{c^2}}}\Rightarrow u'x'=zero$$

$$ux=v \qquad u'x'=zero \qquad 21.01$$

(1.17) $$dt'=dt\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}=dt\sqrt{1+\frac{v^2}{c^2}-\frac{2vv}{c^2}}\Rightarrow dt'=dt\sqrt{1-\frac{v^2}{c^2}}$$

(1.22) $$dt=dt'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}=dt'\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'(0)}{c^2}}\Rightarrow dt=dt'\sqrt{1+\frac{v'^2}{c^2}}$$

$$dt'=dt\sqrt{1-\frac{v^2}{c^2}} \qquad dt=dt'\sqrt{1+\frac{v'^2}{c^2}} \qquad 21.02$$

$$\sqrt{1-\frac{v^2}{c^2}}\sqrt{1+\frac{v'^2}{c^2}}=1 \qquad 21.03$$

$$v=\frac{v'}{\sqrt{1+\frac{v'^2}{c^2}}} \qquad v'=\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} \qquad 21.04$$

$$dt>dt' \qquad v<v' \qquad vdt=v'dt' \qquad 21.05$$

(1.33) $$\vec{v}=\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'u'x'}{c^2}}}=\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}+\frac{2v'(0)}{c^2}}}\Rightarrow\vec{v}=\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}$$

$$(1.34)\quad \vec{v}'=\frac{-\vec{v}}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vux}{c^2}}}=\frac{-\vec{v}}{\sqrt{1+\frac{v^2}{c^2}-\frac{2vv}{c^2}}}\Rightarrow\vec{v}'=\frac{-\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\vec{v}=\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}\qquad -\vec{v}'=\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\qquad 21.06$$

$$\vec{r}=r\hat{r}=-\vec{r}'\qquad \vec{r}'=-r\hat{r}=-\vec{r}\qquad |\vec{r}|=|\vec{r}'|=r\qquad 21.07$$

$$d\vec{r}=dr\hat{r}+rd\hat{r}=-d\vec{r}'\qquad d\vec{r}'=-dr\hat{r}-rd\hat{r}=-d\vec{r}\qquad 21.08$$

$$\hat{r}d\vec{r}=dr\hat{r}\hat{r}+r\hat{r}d\hat{r}=dr\qquad \hat{r}d\vec{r}'=-dr\hat{r}\hat{r}-r\hat{r}d\hat{r}=-dr\qquad 21.09$$

$$\vec{v}=\frac{d\vec{r}}{dt}=\frac{d(r\hat{r})}{dt}=\frac{dr}{dt}\hat{r}+r\frac{d\phi}{dt}\hat{\phi}\qquad v^2=\vec{v}\vec{v}=\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\phi}{dt}\right)^2\qquad 21.10$$

$$\vec{v}'=\frac{d\vec{r}'}{dt'}=\frac{d(-r\hat{r})}{dt'}=-\left(\frac{dr}{dt'}\hat{r}+r\frac{d\phi}{dt'}\hat{\phi}\right)\qquad v'^2=\vec{v}'\vec{v}'=\left(\frac{dr}{dt'}\right)^2+\left(r\frac{d\phi}{dt'}\right)^2\qquad 21.11$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d^2\vec{r}}{dt^2}=\frac{d^2(r\hat{r})}{dt^2}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\phi}{dt}\right)^2\right]\hat{r}+\left(2\frac{dr}{dt}\frac{d\phi}{dt}+r\frac{d^2\phi}{dt^2}\right)\hat{\phi}\qquad 21.12$$

$$\vec{a}'=\frac{d\vec{v}'}{dt'}=\frac{d^2\vec{r}'}{dt'^2}=\frac{d^2(-r\hat{r})}{dt'^2}=-\left[\frac{d^2r}{dt'^2}-r\left(\frac{d\phi}{dt'}\right)^2\right]\hat{r}-\left(2\frac{dr}{dt'}\frac{d\phi}{dt'}+r\frac{d^2\phi}{dt'^2}\right)\hat{\phi}\qquad 21.13$$

$$\vec{v}=\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}\qquad 21.06$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d}{dt}\left(\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}\right)=\frac{dt'}{dt}\frac{d}{dt'}\left(\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}\right)=\sqrt{1-\frac{v^2}{c^2}}\frac{d}{dt'}\left(\frac{-\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}\right)\qquad 21.50$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\sqrt{1-\frac{v^2}{c^2}}\frac{-1}{\left(1+\frac{v'^2}{c^2}\right)}\left[\sqrt{1+\frac{v'^2}{c^2}}\frac{d\vec{v}'}{dt'}-\vec{v}'\frac{d}{dt'}\left(\sqrt{1+\frac{v'^2}{c^2}}\right)\right]$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\sqrt{1-\frac{v^2}{c^2}}\frac{-1}{\left(1+\frac{v'^2}{c^2}\right)}\left[\sqrt{1+\frac{v'^2}{c^2}}\frac{d\vec{v}'}{dt'}-\vec{v}'\frac{1}{2}\left(1+\frac{v'^2}{c^2}\right)^{\frac{1}{2}-\frac{2}{2}=\frac{-1}{2}}\left(\frac{2v'}{c^2}\frac{dv'}{dt'}\right)\right]$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\sqrt{1-\frac{v^2}{c^2}}\frac{-1}{\left(1+\frac{v'^2}{c^2}\right)}\left(\sqrt{1+\frac{v'^2}{c^2}}\frac{d\vec{v}'}{dt'}-\frac{1}{\sqrt{1+\frac{v'^2}{c^2}}}v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right)$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\sqrt{1-\frac{v^2}{c^2}}\,\frac{-1}{\left(1+\frac{v'^2}{c^2}\right)}\left(\frac{1}{\sqrt{1+\frac{v'^2}{c^2}}}\sqrt{1+\frac{v'^2}{c^2}}\,\frac{d\vec{v}'}{dt'}\sqrt{1+\frac{v'^2}{c^2}}-\frac{1}{\sqrt{1+\frac{v'^2}{c^2}}}\,v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right)$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\sqrt{1-\frac{v^2}{c^2}}\,\frac{-1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'}-v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right]$$

$$m\vec{a}=\frac{m_0\vec{a}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt}=\frac{-m_0}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'}-v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right]$$

$$\vec{F}=m\vec{a}=\frac{m_0\vec{a}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt} \qquad 21.51$$

$$\vec{F}'=\frac{-m_0}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'}-v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right] \qquad 21.52$$

$$\vec{F}=m\vec{a}=\frac{m_0\vec{a}}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt}=\vec{F}'=\frac{-m_0}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'}-v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right] \qquad 21.53$$

$$E_k=\int\vec{F}\cdot d\vec{r}=\int\vec{F}'\,\left(-d\vec{r}'\right)=\int\frac{-k}{r^2}\hat{r}\left(-d\vec{r}'\right) \qquad 21.54$$

$$E_k=\int\vec{F}\cdot d\vec{r}=\int\vec{F}'\,\left(-d\vec{r}'\right)=\int\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt}d\vec{r}=\int\frac{-m_0}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'}-v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right]\left(-d\vec{r}'\right)=\int\frac{-k}{r^2}\hat{r}\left(-d\vec{r}'\right) \qquad 21.55$$

$$E_k=\int\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}d\vec{v}\frac{d\vec{r}}{dt}=\int\frac{m_0}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)d\vec{v}'\frac{d\vec{r}'}{dt'}-v'dv'\frac{d\vec{r}'}{dt'}\frac{\vec{v}'}{c^2}\right]=\int\frac{k}{r^2}\hat{r}\,d\vec{r}'$$

$$E_k=\int\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}d\vec{v}\vec{v}=\int\frac{m_0}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)d\vec{v}'\vec{v}'-v'dv'\frac{\vec{v}'\vec{v}'}{c^2}\right]=\int\frac{-k}{r^2}dr$$

$$E_k=\int\frac{m_0 v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}=\int\frac{m_0}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)d\vec{v}'\vec{v}'-v'dv'\frac{v'^2}{c^2}\right]=\int\frac{-k}{r^2}dr$$

$$E_k=\int\frac{m_o v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}=\int\frac{m_o v'dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left(1+\frac{v'^2}{c^2}-\frac{v'^2}{c^2}\right)=\int\frac{-k}{r^2}dr$$

$$E_k=\int\frac{m_o v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}=\int\frac{m_o v'dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=\int\frac{-k}{r^2}dr \qquad dE_k=\frac{m_o v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_o v'dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=\frac{-k}{r^2}dr \quad 21.56$$

$$E_k=-m_o c^2\sqrt{1-\frac{v^2}{c^2}}=\frac{-m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{k}{r}+constante \quad 21.57$$

$$E_R=-m_o c^2\sqrt{1-\frac{v^2}{c^2}}-\frac{k}{r}=constante \qquad E_R=\frac{-m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}}-\frac{k}{r}=constante \quad 21.58$$

$$E_R=\frac{-m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}}-\frac{k}{r}=-m_o c^2+\frac{m_o v'^2}{2}-\frac{k}{r} \qquad E_R=\frac{-m_o c^2}{\sqrt{1+\frac{(0)^2}{c^2}}}-\frac{k}{\infty}=-m_o c^2 \quad 21.59$$

$$\frac{-1}{\sqrt{1+\frac{v'^2}{c^2}}}=\frac{E_R}{m_o c^2}+\frac{k}{m_o c^2}\frac{1}{r} \quad 21.60$$

$$H=\frac{E_R}{m_o c^2} \qquad A=\frac{k}{m_o c^2}=\frac{GM_o m_o}{m_o c^2}=\frac{GM_o}{c^2} \quad 21.61$$

$$\frac{-1}{\sqrt{1+\frac{v'^2}{c^2}}}=H+A\frac{1}{r} \qquad \frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=-\left(H+A\frac{1}{r}\right)^3 \quad 21.62$$

$$\vec{L}'=\vec{r}'\times\vec{v}'=-r\hat{r}\times\left[-\left(\frac{dr}{dt'}\hat{r}+r\frac{d\phi}{dt'}\hat{\phi}\right)\right]=r^2\frac{d\phi}{dt'}(\hat{r}\times\hat{\phi})=r^2\frac{d\phi}{dt'}\hat{k} \quad 21.63$$

$$\vec{L}'=\vec{r}'\times\vec{v}'=-\vec{r}\times\frac{-\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}=r\hat{r}\times\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[\left(\frac{dr}{dt}\hat{r}+r\frac{d\phi}{dt}\hat{\phi}\right)\right]=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}r^2\frac{d\phi}{dt}(\hat{r}\times\hat{\phi})=r^2\frac{d\phi}{dt'}\hat{k} \quad 21.63$$

$$\vec{L}'=r^2\frac{d\phi}{dt'}\vec{k}=L'\hat{k} \qquad L'=r^2\frac{d\phi}{dt'} \quad 21.64$$

$$dE_k=\frac{m_o v\,dv}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m_o v'dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=\frac{-k}{r^2}dr=\frac{k}{r^2}\hat{r}d\vec{r}' \quad 21.56$$

$$\frac{dE_k}{dt'}=\vec{F}'\vec{v}'=\frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}v'\frac{dv'}{dt'}=\frac{k}{r^2}\hat{r}\frac{d\vec{r}'}{dt'}=\frac{k}{r^2}\hat{r}\vec{v}'$$

$$\vec{F}'=\frac{m_o\vec{a}'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}=\frac{k}{r^2}\hat{r} \qquad 21.65$$

$$\vec{F}'=\frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left\{-\left[\frac{d^2r}{dt'^2}-r\left(\frac{d\phi}{dt'}\right)^2\right]\hat{r}-\left(2\frac{dr}{dt'}\frac{d\phi}{dt'}+r\frac{d^2\phi}{dt'^2}\right)\hat{\phi}\right\}=\frac{k}{r^2}\hat{r} \qquad 21.66$$

$$F'_{\hat{\phi}}=\frac{-m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left(2\frac{dr}{dt'}\frac{d\phi}{dt'}+r\frac{d^2\phi}{dt'^2}\right)\hat{\phi}=zero \qquad 21.67$$

$$F'_{\hat{r}}=\frac{-m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{d^2r}{dt'^2}-r\left(\frac{d\phi}{dt'}\right)^2\right]\hat{r}=\frac{k}{r^2}\hat{r} \qquad 21.68$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{d^2r}{dt'^2}-r\left(\frac{d\phi}{dt'}\right)^2\right]\hat{r}=\frac{-GM_o}{r^2}\hat{r}$$

$$\frac{d\phi}{dt'}=\frac{L'}{r^2} \qquad \frac{dr}{dt'}=-L'\frac{dw}{d\phi} \qquad \frac{d^2r}{dt'^2}=\frac{-L'^2}{r^2}\frac{d^2w}{d\phi^2} \qquad \frac{d^2\phi}{dt'^2}=\frac{2L'^2}{r^3}\frac{dw}{d\phi} \qquad 21.69$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{-L'^2}{r^2}\frac{d^2w}{d\phi^2}-r\left(\frac{L'}{r^2}\right)^2\right]=\frac{-GM_o}{r^2}$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left(\frac{-L'^2}{r^2}\frac{d^2w}{d\phi^2}-\frac{L'^2}{r^3}\right)=\frac{-GM_o}{r^2}$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)\left(\frac{-L'^2}{r^2}\right)=\frac{-GM_o}{r^2}$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{GM_o}{L'^2} \qquad 21.70$$

$$-\left(H+A\frac{1}{r}\right)^3\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{GM_o}{L'^2}$$ 21.71

§25 Logarithmic Spiral continued

$$-\left(H+A\frac{1}{r}\right)^3\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{GM_o}{L'^2}$$ 21.71

$$\left(H+A\frac{1}{r}\right)^3\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=\frac{-GM_o}{L'^2}$$

$$\left(H+A\frac{1}{r}\right)^3\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)=-B \qquad H=\frac{E_R}{m_oc^2} \qquad A=\frac{GM_o}{c^2} \qquad B=\frac{GM_o}{L'^2}$$

$$\left(H+A\frac{1}{r}\right)^3\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)+B=zero$$

$$\left(H^3+3H^2A\frac{1}{r}+3HA^2\frac{1}{r^2}+A^3\frac{1}{r^3}\right)\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)+B=zero$$

$$H^3+3H^2A\frac{1}{r}+3HA^2\frac{1}{r^2}+A^3\frac{1}{r^3}\cong H^3+3H^2A\frac{1}{r} \qquad 3HA^2\frac{1}{r^2}+A^3\frac{1}{r^3}\cong zero$$

$$\left(H^3+3AH^2\frac{1}{r}\right)\left(\frac{d^2w}{d\phi^2}+\frac{1}{r}\right)+B=zero$$

$$\left(H^3+3AH^2w\right)\left(\frac{d^2w}{d\phi^2}+w\right)+B=zero$$

$$H^3\frac{d^2w}{d\phi^2}+H^3w+3AH^2\frac{d^2w}{d\phi^2}w+3AH^2w^2+B=zero$$

$$w=\frac{1}{r}=\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)] \qquad \frac{dw}{d\phi}=\frac{-Q\sin(\phi Q)}{D} \qquad \frac{d^2w}{d\phi^2}=\frac{-Q^2\cos(\phi Q)}{D}$$ 21.38

The first hypothesis to obtain a particular solution of the differential equation is to assume the infinite radius $r=\infty$, thus obtaining:

$$w=\frac{1}{r=\infty}=\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]=zero\Rightarrow\varepsilon\cos(\phi Q)=-1 \qquad \frac{d^2w}{d\phi^2}=\frac{-Q^2\cos(\phi Q)}{D}=\frac{-Q^2\varepsilon\cos(\phi Q)}{\varepsilon D}=\frac{Q^2}{\varepsilon D}$$

$$H^3\frac{d^2w}{d\phi^2}+H^3w+3AH^2\frac{d^2w}{d\phi^2}w+3AH^2w^2+B=zero$$

$$w=zero \qquad \frac{d^2w}{d\phi^2}=\frac{Q^2}{\varepsilon D} \qquad H=\frac{E_R}{m_oc^2}=\frac{-m_oc^2}{m_oc^2}=-1$$

$$(-1)^3\left(\frac{Q^2}{\varepsilon D}\right)+(-1)^3(zero)+3A(-1)^2\left(\frac{Q^2}{\varepsilon D}\right)(zero)+3A(-1)^2(zero)^2+B=zero$$

$$-\left(\frac{Q^2}{\varepsilon D}\right)+B=zero \qquad -\frac{\varepsilon DQ^2}{\varepsilon D}+\varepsilon DB=zero$$

$$-Q^2+1=zero \qquad Q^2=1$$

This result shows that in infinity the influence of the central mass is zero $M_o=zero$.

The second hypothesis to obtain another particular solution of the differential equation is obtained by observing that the angle (ϕQ) of the equation $\varepsilon\cos(\phi Q)=-1$ indicates the direction of the infinite radius $r=\infty$ where the influence of the central mass is zero $M_o=zero$ and $Q^2=1$ therefore the direction of the center of mass is given by the angle $(\phi Q+\pi)$ that replaced in the equation $\varepsilon\cos(\phi Q)=-1$ results in the new equation $\varepsilon\cos(\phi Q+\pi)=-1$ that indicates direction opposite the direction of the infinite radius which is the direction of the center of mass.

$$\varepsilon\cos(\phi Q+\pi)=-1 \qquad \cos(\phi Q+\pi)=-\cos(\phi Q) \qquad \varepsilon[-\cos(\phi Q)]=-1 \qquad \varepsilon\cos(\phi Q)=1$$

$$w=\frac{1}{r}=\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]=\frac{1}{\varepsilon D}(1+1)=\frac{2}{\varepsilon D} \qquad \frac{d^2w}{d\phi^2}=\frac{-Q^2\cos(\phi Q)}{D}=\frac{-Q^2\varepsilon\cos(\phi Q)}{\varepsilon D}=\frac{-Q^2}{\varepsilon D}$$

$$w=\frac{2}{\varepsilon D} \qquad \frac{d^2w}{d\phi^2}=\frac{-Q^2}{\varepsilon D} \qquad H=\frac{E_R}{m_oc^2}=\frac{-m_oc^2}{m_oc^2}=-1$$

$$H^3\frac{d^2w}{d\phi^2}+H^3w+3AH^2\frac{d^2w}{d\phi^2}w+3AH^2w^2+B=zero$$

$$(-1)^3\left(\frac{-Q^2}{\varepsilon D}\right)+(-1)^3\left(\frac{2}{\varepsilon D}\right)+3A(-1)^2\left(\frac{-Q^2}{\varepsilon D}\right)\left(\frac{2}{\varepsilon D}\right)+3A(-1)^2\left(\frac{2}{\varepsilon D}\right)^2+B=zero$$

$$-\left(\frac{-Q^2}{\varepsilon D}\right)-\left(\frac{2}{\varepsilon D}\right)+3A\left(\frac{-Q^2}{\varepsilon D}\right)\left(\frac{2}{\varepsilon D}\right)+3A\left(\frac{2}{\varepsilon D}\right)^2+B=zero$$

$$\frac{Q^2}{\varepsilon D}-\frac{2}{\varepsilon D}-3A\frac{Q^2}{\varepsilon D}\frac{2}{\varepsilon D}+3A\frac{4}{\varepsilon^2D^2}+B=zero$$

$$\frac{Q^2}{\varepsilon D}-\frac{2}{\varepsilon D}-\frac{6AQ^2}{\varepsilon^2D^2}+\frac{12A}{\varepsilon^2D^2}+B=zero$$

$$\frac{\varepsilon DQ^2}{\varepsilon D}-\frac{2\varepsilon D}{\varepsilon D}-\frac{\varepsilon D6AQ^2}{\varepsilon^2D^2}+\frac{\varepsilon D12A}{\varepsilon^2D^2}+\varepsilon DB=zero \qquad \varepsilon DB=\frac{\varepsilon DGM_o}{L'^2}=\frac{\varepsilon DGM_o}{\varepsilon DGM_o}=1$$

$$Q^2-2-\frac{6AQ^2}{\varepsilon D}+\frac{12A}{\varepsilon D}+1=zero$$

$$Q^2-1-\frac{6AQ^2}{\varepsilon D}+\frac{12A}{\varepsilon D}=zero$$

$$Q^2-\frac{6AQ^2}{\varepsilon D}=1-\frac{12A}{\varepsilon D} \qquad Q^2=\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}$$

Applying the results of the second hypothesis in the differential equation:

$$H^3\frac{d^2w}{d\phi^2}+H^3w+3AH^2\frac{d^2w}{d\phi^2}w+3AH^2w^2+B=zero$$

$$w=\frac{1}{r}=\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)] \qquad \frac{dw}{d\phi}=\frac{-Q\sin(\phi Q)}{D} \qquad \frac{d^2w}{d\phi^2}=\frac{-Q^2\cos(\phi Q)}{D} \tag{21.38}$$

$$H^3\left[\frac{-Q^2\cos(\phi Q)}{D}\right]+H^3\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]+3AH^2\left[\frac{-Q^2\cos(\phi Q)}{D}\right]\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]+$$
$$+3AH^2\left\{\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]\right\}^2+B=zero$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D}+H^3\frac{1}{\varepsilon D}+H^3\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q)+3AH^2\left[\frac{-Q^2\cos(\phi Q)}{D}\right]\frac{1}{\varepsilon D}+3AH^2\left[\frac{-Q^2\cos(\phi Q)}{D}\right]\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q)+$$
$$+3AH^2\left\{\frac{1}{\varepsilon^2D^2}\left[1+2\varepsilon\cos(\phi Q)+\varepsilon^2\cos^2(\phi Q)\right]\right\}+B=zero$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D}+\frac{H^3}{\varepsilon D}+H^3\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-3AH^2Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{\varepsilon^2D^2}\left[1+2\varepsilon\cos(\phi Q)+\varepsilon^2\cos^2(\phi Q)\right]+B=zero$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D}+\frac{H^3}{\varepsilon D}+H^3\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-3AH^2Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{\varepsilon^2D^2}+\frac{3AH^2}{\varepsilon^2D^2}2\varepsilon\cos(\phi Q)+\frac{3AH^2}{\varepsilon^2D^2}\varepsilon^2\cos^2(\phi Q)+B=zero$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D}+\frac{H^3}{\varepsilon D}+H^3\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-3AH^2Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{\varepsilon^2D^2}+\frac{6AH^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+3AH^2\frac{\cos^2(\phi Q)}{D^2}+B=zero$$

$$\frac{-H^3Q^2}{3AH^2}\frac{\cos(\phi Q)}{D}+\frac{H^3}{3AH^2\varepsilon D}+\frac{H^3}{3AH^2}\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{3AH^2\varepsilon D}\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{3AH^2}\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{3AH^2\varepsilon^2D^2}+\frac{6AH^2}{3AH^2\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{3AH^2}{3AH^2}\frac{\cos^2(\phi Q)}{D^2}+\frac{B}{3AH^2}=zero$$

$$\frac{-HQ^2}{3A}\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{H}{3A}\frac{\cos(\phi Q)}{D}-\frac{Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{1}{\varepsilon^2D^2}+\frac{2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{\cos^2(\phi Q)}{D^2}+\frac{B}{3AH^2}=zero$$

$$\frac{\cos^2(\phi Q)}{D^2}-Q^2\frac{\cos^2(\phi Q)}{D^2}-\frac{HQ^2}{3A}\frac{\cos(\phi Q)}{D}+\frac{H}{3A}\frac{\cos(\phi Q)}{D}-\frac{Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+$$

$$+\frac{2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{B}{3AH^2}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-\frac{HQ^2}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{B}{3AH^2}=\text{zero}$$

$$H=\frac{E_R}{m_o c^2}=\frac{-m_o c^2}{m_o c^2}=-1$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-\frac{(-1)Q^2}{3A}+\frac{(-1)}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{(-1)}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{B}{3A(-1)^2}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}-\frac{1}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{B}{3A}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}-\frac{1}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{\varepsilon DB}{3A\varepsilon D}=\text{zero}$$

$$\varepsilon DB=\frac{\varepsilon DGM_o}{L'^2}=\frac{\varepsilon DGM_o}{\varepsilon DGM_o}=1$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}-\frac{1}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{1}{3A\varepsilon D}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=\text{zero}$$

$$\text{zero}<r(\phi Q)<\infty\to M_o\neq\text{zero}\to Q=\frac{\sqrt{1-\frac{12A}{\varepsilon D}}}{\sqrt{1-\frac{6A}{\varepsilon D}}}$$

$$\left[1-\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)\right]\frac{\cos^2(\phi Q)}{D^2}+\left[\frac{1}{3A}\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)-\frac{1}{3A}-\frac{1}{\varepsilon D}\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)+\frac{2}{\varepsilon D}\right]\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=\text{zero}$$

$$\left[1-\frac{6A}{\varepsilon D}-\left(1-\frac{12A}{\varepsilon D}\right)\right]\frac{\cos^2(\phi Q)}{D^2}+\left[\frac{1}{3A}\left(1-\frac{12A}{\varepsilon D}\right)-\frac{1}{3A}\left(1-\frac{6A}{\varepsilon D}\right)-\frac{1}{\varepsilon D}\left(1-\frac{12A}{\varepsilon D}\right)+\frac{2}{\varepsilon D}\left(1-\frac{6A}{\varepsilon D}\right)\right]\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}\left(1-\frac{6A}{\varepsilon D}\right)=\text{zero}$$

$$\left(1-\frac{6A}{\varepsilon D}-1+\frac{12A}{\varepsilon D}\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{1}{3A}-\frac{1}{3A}\frac{12A}{\varepsilon D}-\frac{1}{3A}+\frac{1}{3A}\frac{6A}{\varepsilon D}-\frac{1}{\varepsilon D}+\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}+\frac{2}{\varepsilon D}-\frac{2}{\varepsilon D}\frac{6A}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}-\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}=\text{zero}$$

$$\left(\frac{6A}{\varepsilon D}\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-\frac{1}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}-\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}=\text{zero}$$

$$\frac{\cos(\phi Q)}{D}=\frac{-\left(-\frac{1}{\varepsilon D}\right)\pm\sqrt{\left(-\frac{1}{\varepsilon D}\right)^{2}-4\frac{6A}{\varepsilon D}\left(\frac{1}{\varepsilon^{2}D^{2}}-\frac{1}{\varepsilon^{2}D^{2}}\frac{6A}{\varepsilon D}\right)}}{2\frac{6A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\sqrt{\frac{1}{\varepsilon^{2}D^{2}}-\frac{24A}{\varepsilon D}\left(\frac{1}{\varepsilon^{2}D^{2}}-\frac{1}{\varepsilon^{2}D^{2}}\frac{6A}{\varepsilon D}\right)}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\sqrt{\frac{1}{\varepsilon^{2}D^{2}}-\frac{24A}{\varepsilon D}\frac{1}{\varepsilon^{2}D^{2}}+\frac{24A}{\varepsilon D}\frac{1}{\varepsilon^{2}D^{2}}\frac{6A}{\varepsilon D}}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\sqrt{1-\frac{24A}{\varepsilon D}+\frac{24A}{\varepsilon D}\frac{6A}{\varepsilon D}}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\sqrt{1-2\frac{12A}{\varepsilon D}+\frac{144A^{2}}{\varepsilon^{2}D^{2}}}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\sqrt{\left(1-\frac{12A}{\varepsilon D}\right)^{2}}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\left(1-\frac{12A}{\varepsilon D}\right)}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\left(\frac{1}{\varepsilon D}-\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}\right)}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}-\left(\frac{1}{\varepsilon D}-\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}\right)}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}-\frac{1}{\varepsilon D}+\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{1}{\varepsilon D}$$

Where applying the result of the second hypothesis $\varepsilon\cos(\phi Q)=1\Rightarrow\cos(\phi Q)=\frac{1}{\varepsilon}$:

$$\frac{1}{\varepsilon}\frac{1}{D}=\frac{1}{\varepsilon D}$$

That it is an identity demonstrating that the result of the second hypothesis is correct.

$$Q^2=\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\approx 1-\frac{6A}{\varepsilon D} \qquad Q^2=1-\frac{6A}{\varepsilon D} \qquad A=\frac{GM_o}{c^2}$$

$$\varepsilon D=a(1-\varepsilon^2)=57.909.227.000,00\left[1-(0,20563593)^2\right]=55.460.469.568,40$$

$$A=\frac{GM_o}{c^2}=\frac{6,6740831\cdot 10^{-11}\cdot 1,9891\cdot 10^{30}}{(2,99792458\cdot 10^{8})^2}=1.477,08953542$$

$$Q=\sqrt{\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}}=0,9999999201 \qquad Q=\sqrt{1-\frac{6A}{\varepsilon D}}=0,9999999201$$

$1,276.789.102.53^{-14}$

$$\phi Q=1.296.000,00\Rightarrow\phi=\frac{1.296.000,00}{Q}$$

$Q<1$ Advance $Q>1$ Retrocess

$$\Delta\phi=\left(\frac{1}{Q}-1\right)1.296.000,00$$

$\Delta\phi>zero$ Advance $\Delta\phi<zero$ Retrocess

$$\Delta\phi=\left[\frac{1}{\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)^{\frac{1}{2}}}-1\right]1.296.000,00=0,103549893544$$

$$\Delta\phi=\left[\frac{1}{\left(1-\frac{6A}{\varepsilon D}\right)^{\frac{1}{2}}}-1\right]1.296.000,00=0,103549876997$$

$$N=100\frac{PT}{PM}=100\frac{365,256363004}{87,969}=415,210316139$$

$$\sum\Delta\phi=\Delta\phi N=0,103549893544\text{ x }415,210316139=42,9949840347''$$

$$\sum\Delta\phi=\Delta\phi N=0,103549876997\text{ x }415,210316139=42,9949771642''$$

Newtonian Energy E_N

$$E_N=\frac{m_o u^2}{2}-\frac{k}{r}$$

$$u^2=\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\phi}{dt}\right)^2=\left(\frac{dr}{dt}\right)^2+\frac{L^2}{r^2}$$

$$E_N=\frac{m_o}{2}\left[\left(\frac{dr}{dt}\right)^2+\frac{L^2}{r^2}\right]-\frac{k}{r}$$

$$\frac{2E_N}{m_o}=\left(\frac{dr}{dt}\right)^2+\frac{L^2}{r^2}-\frac{2k}{m_o}\frac{1}{r}$$

$$\left(\frac{dr}{dt}\right)^2+\frac{L^2}{r^2}-\frac{2k}{m_o}\frac{1}{r}-\frac{2E_N}{m_o}=zero$$

$$\frac{d\phi}{dt}=\frac{L}{r^2} \qquad \frac{dr}{dt}=-L\frac{dw}{d\phi} \qquad \frac{d^2r}{dt^2}=\frac{-L^2}{r^2}\frac{d^2w}{d\phi^2} \qquad \frac{d^2\phi}{dt^2}=\frac{2L^2}{r^3}\frac{dw}{d\phi}$$

$$\left(-L\frac{dw}{d\phi}\right)^2+\frac{L^2}{r^2}-\frac{2k}{m_o}\frac{1}{r}-\frac{2E_N}{m_o}=zero$$

$$\left(\frac{dw}{d\phi}\right)^2+\frac{1}{r^2}-\frac{2k}{m_oL^2}\frac{1}{r}-\frac{2E_N}{m_oL^2}=zero$$

$$\left(\frac{dw}{d\phi}\right)^2+\frac{1}{r^2}-\frac{2k}{m_oL^2}\frac{1}{r}-\frac{2E_N}{m_oL^2}=zero$$

$$\left(\frac{dw}{d\phi}\right)^2+w^2-\frac{2k}{m_oL^2}w-\frac{2E_N}{m_oL^2}=zero$$

$$x=\frac{2k}{m_oL^2} \qquad y=\frac{2E_N}{m_oL^2}$$

$$\left(\frac{dw}{d\phi}\right)^2+w^2-xw-y=zero$$

$$w=\frac{1}{r}=\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right] \qquad \frac{dw}{d\phi}=\frac{-Q\,sen(\phi Q)}{D} \qquad \frac{d^2w}{d\phi^2}=\frac{-Q^2\cos(\phi Q)}{D}$$

$$\left[\frac{-Q\,sen(\phi Q)}{D}\right]^2+\left\{\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right]\right\}^2-x\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right]-y=zero$$

$$\frac{Q^2}{D^2}\left[1-\cos^2(\phi Q)\right]+\frac{1}{\varepsilon^2D^2}\left[1+2\varepsilon\cos(\phi Q)+\varepsilon^2\cos^2(\phi Q)\right]-x\frac{1}{\varepsilon D}-x\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q)-y=zero$$

$$\frac{Q^2}{D^2}-\frac{Q^2}{D^2}\cos^2(\phi Q)+\frac{1}{\varepsilon^2 D^2}+\frac{1}{\varepsilon^2 D^2}2\varepsilon\cos(\phi Q)+\frac{1}{\varepsilon^2 D^2}\varepsilon^2\cos^2(\phi Q)-\frac{x}{\varepsilon D}-x\frac{\cos(\phi Q)}{D}-y=zero$$

$$\frac{Q^2}{D^2}-Q^2\frac{\cos^2(\phi Q)}{D^2}+\frac{1}{\varepsilon^2 D^2}+\frac{2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{\cos^2(\phi Q)}{D^2}-\frac{x}{\varepsilon D}-x\frac{\cos(\phi Q)}{D}-y=zero$$

$$\frac{\cos^2(\phi Q)}{D^2}-Q^2\frac{\cos^2(\phi Q)}{D^2}+\frac{2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-x\frac{\cos(\phi Q)}{D}+\frac{Q^2}{D^2}+\frac{1}{\varepsilon^2 D^2}-\frac{x}{\varepsilon D}-y=zero$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{2}{\varepsilon D}-x\right)\frac{\cos(\phi Q)}{D}+\frac{Q^2}{D^2}+\frac{1}{\varepsilon^2 D^2}-\frac{x}{\varepsilon D}-y=zero$$

Newtonian Energy E_N

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(x-\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{Q^2}{D^2}+\frac{1}{\varepsilon^2 D^2}-\frac{x}{\varepsilon D}-y=zero$$

$$r=\infty\rightarrow Q=1\rightarrow w=\frac{1}{r=\infty}=\frac{1}{\varepsilon D}\left[1+\varepsilon\cos(\phi Q)\right]=\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon D}=zero$$

$$\left(1-Q^2\right)\left(-\frac{1}{\varepsilon D}\right)\left(-\frac{1}{\varepsilon D}\right)+\left(x-\frac{2}{\varepsilon D}\right)\left(-\frac{1}{\varepsilon D}\right)+\frac{Q^2}{D^2}+\frac{1}{\varepsilon^2 D^2}-\frac{x}{\varepsilon D}-y=zero$$

$$\left(1-Q^2\right)\left(\frac{1}{\varepsilon^2 D^2}\right)-\frac{x}{\varepsilon D}+\frac{2}{\varepsilon^2 D^2}+\frac{Q^2}{D^2}+\frac{1}{\varepsilon^2 D^2}-\frac{x}{\varepsilon D}-y=zero$$

$$\frac{1}{\varepsilon^2 D^2}-\frac{Q^2}{\varepsilon^2 D^2}-\frac{x}{\varepsilon D}+\frac{2}{\varepsilon^2 D^2}+\frac{Q^2}{D^2}+\frac{1}{\varepsilon^2 D^2}-\frac{x}{\varepsilon D}-y=zero$$

$$-\frac{Q^2}{\varepsilon^2 D^2}+\frac{Q^2}{D^2}+\frac{4}{\varepsilon^2 D^2}-\frac{2x}{\varepsilon D}-y=zero \qquad Q^2=1$$

$$-\frac{1}{\varepsilon^2 D^2}+\frac{1}{D^2}+\frac{4}{\varepsilon^2 D^2}-\frac{2x}{\varepsilon D}-y=zero$$

$$-\frac{\varepsilon^2 D^2}{\varepsilon^2 D^2}+\frac{\varepsilon^2 D^2}{D^2}+\frac{4\varepsilon^2 D^2}{\varepsilon^2 D^2}-\frac{2x\varepsilon^2 D^2}{\varepsilon D}-\varepsilon^2 D^2 y=zero$$

$$-1+\varepsilon^2+4-2x\varepsilon D-\varepsilon^2 D^2 y=zero$$

$$x=\frac{2}{\varepsilon D} \qquad y=\frac{2E_N}{m_o L^2} \qquad L^2=\varepsilon DGM \qquad \frac{1}{a}=\frac{-1}{\varepsilon D}\left(\varepsilon^2-1\right)$$

$$-1+\varepsilon^2+4-2\frac{2}{\varepsilon D}\varepsilon D-\varepsilon^2 D^2 y=zero \qquad -1+\varepsilon^2-\varepsilon^2 D^2 y=zero$$

$$-1+\varepsilon^2-\varepsilon^2 D^2\frac{2E_N}{m_o L^2}=zero \qquad -1+\varepsilon^2-\varepsilon^2 D^2\frac{2E_N}{m_o\varepsilon DGM_o}=zero$$

$$-1+\varepsilon^2-\varepsilon D\frac{2E_N}{GM_o m_o}=zero \qquad \frac{1}{\varepsilon D}\left(\varepsilon^2-1\right)=\frac{2E_N}{k} \qquad E_N=\frac{-k}{2a}$$

Let us start from the equation expressing the equilibrium of forces:

$$\vec{F'} = \frac{m_o \vec{a'}}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = \frac{k}{r^2}\hat{r} \qquad 21.65$$

On the right side we have the gravitational force $\frac{k}{r^2}\hat{r}$ defined by Newton, on the left side we have the physical description of Force $\vec{F'} = \frac{m_O \vec{a'}}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}$ of the Undulating Relativity.

The physical properties of equation 21.65 require its validity when its radius varies from a radius greater than zero to an infinite radius, so the radius varies from $zero < r \leq \infty$, and so we have two distinct boundary conditions. The first boundary condition is when the radius is infinite $r = \infty$ and the gravitational force is zero, which means that the particle is at rest with $v' = zero$ and $\vec{a'} = zero$ and the second boundary condition is when the radius is greater which is zero and smaller than infinity $zero < r < \infty$ which means that the particle is in motion due to the influence of a gravitational force 21.65 with $v' \neq zero$ and $\vec{a'} \neq zero$.

In §26 following the calculations is substituted in 21.65, the equality, 21.62, 21.69 and

$$H = \frac{E_R}{m_o c^2} \qquad A = \frac{GM_o}{c^2} \qquad B = \frac{GM_o}{L'^2}$$, more $w = \frac{1}{r}$.

After these substitutions we obtain the differential equation:

$$H^3\frac{d^2w}{d\phi^2} + H^3 w + 3AH^2\frac{d^2w}{d\phi^2}w + 3AH^2 w^2 + B = zero \qquad 27.1$$

This equation has to be valid for the same boundary conditions as equation 21.65, that is, it has to be valid from a radius r greater than zero $(r > zero)$ to an infinite radius $(zero < r \leq \infty)$. Your solution is given by:

$$w = \frac{1}{r} = \frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)] \qquad 27.2$$

Which should cover the two contour conditions already described.

Applying solution 27.2 in differential equation 27.1 we have:

$$H^3\frac{d^2w}{d\phi^2} + H^3 w + 3AH^2\frac{d^2w}{d\phi^2}w + 3AH^2 w^2 + B = zero$$

$$w = \frac{1}{r} = \frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)] \qquad \frac{dw}{d\phi} = \frac{-Q\,sen(\phi Q)}{D} \qquad \frac{d^2w}{d\phi^2} = \frac{-Q^2\cos(\phi Q)}{D} \qquad 21.38$$

$$H^3\left[\frac{-Q^2\cos(\phi Q)}{D}\right] + H^3\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)] + 3AH^2\left[\frac{-Q^2\cos(\phi Q)}{D}\right]\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)] +$$
$$+3AH^2\left\{\frac{1}{\varepsilon D}[1+\varepsilon\cos(\phi Q)]\right\}^2 + B = zero$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D} + H^3\frac{1}{\varepsilon D} + H^3\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q) + 3AH^2\left[\frac{-Q^2\cos(\phi Q)}{D}\right]\frac{1}{\varepsilon D} + 3AH^2\left[\frac{-Q^2\cos(\phi Q)}{D}\right]\frac{1}{\varepsilon D}\varepsilon\cos(\phi Q) +$$
$$+3AH^2\left\{\frac{1}{\varepsilon^2 D^2}\left[1+2\varepsilon\cos(\phi Q)+\varepsilon^2\cos^2(\phi Q)\right]\right\} + B = zero$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D}+\frac{H^3}{\varepsilon D}+H^3\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-3AH^2Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{\varepsilon^2D^2}\left[1+2\varepsilon\cos(\phi Q)+\varepsilon^2\cos^2(\phi Q)\right]+B=\text{zero}$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D}+\frac{H^3}{\varepsilon D}+H^3\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-3AH^2Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{\varepsilon^2D^2}+\frac{3AH^2}{\varepsilon^2D^2}2\varepsilon\cos(\phi Q)+\frac{3AH^2}{\varepsilon^2D^2}\varepsilon^2\cos^2(\phi Q)+B=\text{zero}$$

$$-H^3Q^2\frac{\cos(\phi Q)}{D}+\frac{H^3}{\varepsilon D}+H^3\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-3AH^2Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{\varepsilon^2D^2}+\frac{6AH^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+3AH^2\frac{\cos^2(\phi Q)}{D^2}+B=\text{zero}$$

$$\frac{-H^3Q^2}{3AH^2}\frac{\cos(\phi Q)}{D}+\frac{H^3}{3AH^2\varepsilon D}+\frac{H^3}{3AH^2}\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{3AH^2\varepsilon D}\frac{\cos(\phi Q)}{D}-\frac{3AH^2Q^2}{3AH^2}\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{3AH^2}{3AH^2\varepsilon^2D^2}+\frac{6AH^2}{3AH^2\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{3AH^2}{3AH^2}\frac{\cos^2(\phi Q)}{D^2}+\frac{B}{3AH^2}=\text{zero}$$

$$\frac{-HQ^2}{3A}\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{H}{3A}\frac{\cos(\phi Q)}{D}-\frac{Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}-Q^2\frac{\cos^2(\phi Q)}{D^2}+$$
$$+\frac{1}{\varepsilon^2D^2}+\frac{2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{\cos^2(\phi Q)}{D^2}+\frac{B}{3AH^2}=\text{zero}$$

$$\frac{\cos^2(\phi Q)}{D^2}-Q^2\frac{\cos^2(\phi Q)}{D^2}-\frac{HQ^2}{3A}\frac{\cos(\phi Q)}{D}+\frac{H}{3A}\frac{\cos(\phi Q)}{D}-\frac{Q^2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+$$
$$+\frac{2}{\varepsilon D}\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3AH^2}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-\frac{HQ^2}{3A}+\frac{H}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{H}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3AH^2}=\text{zero}$$

$$H=\frac{E_R}{m_oc^2}=\frac{-m_oc^2}{m_oc^2}=-1$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-\frac{(-1)Q^2}{3A}+\frac{(-1)}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{(-1)}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3A(-1)^2}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}-\frac{1}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{B}{3A}=\text{zero}$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}-\frac{1}{3A\varepsilon D}+\frac{1}{\varepsilon^2D^2}+\frac{\varepsilon DB}{3A\varepsilon D}=\text{zero}$$

$$\varepsilon DB=\frac{\varepsilon DGM_o}{L'^2}=\frac{\varepsilon DGM_o}{\varepsilon DGM_o}=1$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}-\frac{1}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{1}{3A\varepsilon D}=zero$$

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=zero \qquad 27.3$$

This equation must have solution for the same two contour conditions of 21.65.

Solution of 27.3 for the first boundary condition which is when the radius is infinite $r=\infty$, and the gravitational force is zero which means that the particle is at rest and we have $v'=zero$ and $\vec{a'}=zero$.

Applying $Q^2=1$ in 27.3 we get:

$$(1-1^2)\frac{cos^2(\emptyset 1)}{D^2}+\left(\frac{1^2}{3A}-\frac{1}{3A}-\frac{1^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{cos(\emptyset 1)}{D}+\frac{1}{\varepsilon^2D^2}=zero.$$

$$\frac{cos(\emptyset)}{D}+\frac{1}{\varepsilon D}=zero \qquad \varepsilon=\frac{-1}{cos(\emptyset)} \qquad 27.4$$

Equation 27.4 is exactly equal to the result of equation 27.2 when the radius is infinite $r=\infty$, w = zero and Q = 1, as shown in 27.5:

$$w=\frac{1}{r=\infty}=\frac{1}{\varepsilon D}[1+\varepsilon cos(\emptyset Q)]=\frac{1}{\varepsilon D}[1+\varepsilon cos(\emptyset 1)]=\frac{cos(\emptyset)}{D}+\frac{1}{\varepsilon D}=zero \qquad 27.5$$

Therefore in 27.4 we have an exact result that describes how in infinity the eccentricity ε is related to the angle ∅ of the infinite radius of the particle, being $\varepsilon\geq 1$ which means that the motion from infinity will be or parabolic with $\varepsilon=1$ or hyperbolic with $\varepsilon>1$. Note that by definition $\varepsilon>zero$.

Solution of 27.3 for the second boundary condition which is when the radius is greater than zero and less than infinity $zero<r<\infty$ which means that the particle is in motion due to the influence of a gravitational force with $v'\neq zero$ and $\vec{a'}\neq zero$.

Applying $Q=\frac{\sqrt{1-\frac{12A}{\varepsilon D}}}{\sqrt{1-\frac{6A}{\varepsilon D}}}$ in 27.3 we have:

$$\left(1-Q^2\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{Q^2}{3A}-\frac{1}{3A}-\frac{Q^2}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2 D^2}=zero \qquad 27.3$$

$$\left[1-\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)\right]\frac{\cos^2(\phi Q)}{D^2}+\left[\frac{1}{3A}\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)-\frac{1}{3A}-\frac{1}{\varepsilon D}\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)+\frac{2}{\varepsilon D}\right]\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2D^2}=zero$$

$$\left[1-\frac{6A}{\varepsilon D}-\left(1-\frac{12A}{\varepsilon D}\right)\right]\frac{\cos^2(\phi Q)}{D^2}+\left[\frac{1}{3A}\left(1-\frac{12A}{\varepsilon D}\right)-\frac{1}{3A}\left(1-\frac{6A}{\varepsilon D}\right)-\frac{1}{\varepsilon D}\left(1-\frac{12A}{\varepsilon D}\right)+\frac{2}{\varepsilon D}\left(1-\frac{6A}{\varepsilon D}\right)\right]\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2D^2}\left(1-\frac{6A}{\varepsilon D}\right)=zero$$

$$\left(1-\frac{6A}{\varepsilon D}-1+\frac{12A}{\varepsilon D}\right)\frac{\cos^2(\phi Q)}{D^2}+\left(\frac{1}{3A}-\frac{1}{3A}\frac{12A}{\varepsilon D}-\frac{1}{3A}+\frac{1}{3A}\frac{6A}{\varepsilon D}-\frac{1}{\varepsilon D}+\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}+\frac{2}{\varepsilon D}-\frac{2}{\varepsilon D}\frac{6A}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2D^2}-\frac{1}{\varepsilon^2D^2}\frac{6A}{\varepsilon D}=zero$$

$$\left(\frac{6A}{\varepsilon D}\right)\frac{\cos^2(\phi Q)}{D^2}+\left(-\frac{1}{\varepsilon D}\right)\frac{\cos(\phi Q)}{D}+\frac{1}{\varepsilon^2D^2}-\frac{1}{\varepsilon^2D^2}\frac{6A}{\varepsilon D}=zero$$

$$\frac{\cos(\phi Q)}{D}=\frac{-\left(-\frac{1}{\varepsilon D}\right)\pm\sqrt{\left(-\frac{1}{\varepsilon D}\right)^2-4\frac{6A}{\varepsilon D}\left(\frac{1}{\varepsilon^2 D^2}-\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}\right)}}{2\frac{6A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\sqrt{\frac{1}{\varepsilon^2 D^2}-\frac{24A}{\varepsilon D}\left(\frac{1}{\varepsilon^2 D^2}-\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}\right)}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\sqrt{\frac{1}{\varepsilon^2 D^2}-\frac{24A}{\varepsilon D}\frac{1}{\varepsilon^2 D^2}+\frac{24A}{\varepsilon D}\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\sqrt{1-\frac{24A}{\varepsilon D}+\frac{24A}{\varepsilon D}\frac{6A}{\varepsilon D}}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\sqrt{1-2\frac{12A}{\varepsilon D}+\frac{144A^2}{\varepsilon^2 D^2}}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\sqrt{\left(1-\frac{12A}{\varepsilon D}\right)^2}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\frac{1}{\varepsilon D}\left(1-\frac{12A}{\varepsilon D}\right)}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\pm\left(\frac{1}{\varepsilon D}-\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}\right)}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}-\left(\frac{1}{\varepsilon D}-\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}\right)}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}-\frac{1}{\varepsilon D}+\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{\frac{1}{\varepsilon D}\frac{12A}{\varepsilon D}}{\frac{12A}{\varepsilon D}}$$

$$\frac{\cos(\phi Q)}{D}=\frac{1}{\varepsilon D}$$

$-\frac{cos(\emptyset Q)}{D}+\frac{1}{\varepsilon D}=zero \qquad \varepsilon=\frac{1}{cos(\emptyset Q)}$ 27.6

In the theory of conic for hyperbole we have $\varepsilon=\frac{c}{a}$, equating to 27.6 we have $\varepsilon=\frac{c}{a}=\frac{1}{cos(\emptyset Q)}$ This results $a=c.cos(\emptyset Q)$) which is the correct formula, of the greater half axis of hyperbola.

Therefore in 27.6 we have an exact result that describes how in the course of $zero<r<\infty$ the eccentricity ε is related to the angle ∅ of the particle, being $\varepsilon\geq 1$ which means that the motion will be or parabolic with $\varepsilon=1$ or hyperbolic with $\varepsilon>1$. Note that by definition $\varepsilon>zero$

§28 Simplified Periellium Advance

Perihelion Retrogression $\quad \mathbf{Q>1}$

Imagine that the sun and Mercury are two particles, with the Sun being at the origin of a coordinate system and Mercury lying at a point A on the xy plane. The vector radius $\vec{r}=r\hat{r}$ connecting the origin to point A will describe Mercury's motion in the xy plane.

In the description of the movement of the planet Mercury to the observer O' corresponds to the variables with line for the observer O as without line being used a single radius $\vec{r}=r\hat{r}$ and a single coordinate system for both observers.

Time t' is a function of time t that is $t'=t'(t)$ and time t is a function of time t' that is t = t (t ').

$$dt=dt'\sqrt{1+\frac{v'^2}{c^2}} \qquad dt'=dt\sqrt{1-\frac{v^2}{c^2}} \qquad 21.02$$

$$\sqrt{1-\frac{v^2}{c^2}}\sqrt{1+\frac{v'^2}{c^2}}=1 \qquad 21.03$$

$$v'=\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} \qquad v=\frac{v'}{\sqrt{1+\frac{v'^2}{c^2}}} \qquad 21.04$$

$$dt>dt' \qquad v'>v \qquad vdt=v'dt' \qquad 21.05$$

$$\vec{r}=r\hat{r} \qquad d\vec{r}=dr\hat{r}+rd\hat{r} \qquad \hat{r}.d\vec{r}=dr\hat{r}.\hat{r}+r\hat{r}.d\hat{r}=dr \qquad 28.01$$

The radius can be considered a function of time $t'=t'(t)$ ie $\vec{r}=\vec{r}(t')=\vec{r}[t'(t)]$ or it can be considered a function of time $t=t(t')$ ie $\vec{r}=\vec{r}(t)=\vec{r}[t(t')]$.

$$\vec{r}=\vec{r}(t')=\vec{r}[t'(t)] \qquad \vec{r}=\vec{r}(t)=\vec{r}[t(t')] \qquad 28.02$$

$$\vec{v}'=\frac{d\vec{r}}{dt'}=\frac{dr}{dt'}\hat{r}+r\frac{d\emptyset}{dt'}\hat{\emptyset} \qquad \vec{v}=\frac{d\vec{r}}{dt}=\frac{dr}{dt}\hat{r}+r\frac{d\emptyset}{dt}\hat{\emptyset}$$

$$\vec{v}'=\frac{d\vec{r}}{dt'}=\frac{d\vec{r}}{dt}\frac{dt}{dt'}=\frac{d\vec{r}}{dt}\frac{1}{\frac{dt'}{dt}}=\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \vec{v}=\frac{d\vec{r}}{dt}=\frac{d\vec{r}}{dt'}\frac{dt'}{dt}=\frac{d\vec{r}}{dt'}\frac{1}{\frac{dt}{dt'}}=\frac{\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}}$$

$$\vec{v}'=\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \vec{v}=\frac{\vec{v}'}{\sqrt{1+\frac{v'^2}{c^2}}} \qquad 28.03$$

$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d^2\vec{r}}{dt^2}=\frac{d^2(r\hat{r})}{dt^2}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\emptyset}{dt}\right)^2\right]\hat{r}+\left(2\frac{dr}{dt}\frac{d\emptyset}{dt}+r\frac{d^2\emptyset}{dt^2}\right)\hat{\emptyset} \qquad 28.04$$

$$\vec{a}'=\frac{d\vec{v}'}{dt'}=\frac{d^2\vec{r}}{dt'^2}=\frac{d^2(r\hat{r})}{dt'^2}=\left[\frac{d^2r}{dt'^2}-r\left(\frac{d\emptyset}{dt'}\right)^2\right]\hat{r}+\left(2\frac{dr}{dt'}\frac{d\emptyset}{dt'}+r\frac{d^2\emptyset}{dt'^2}\right)\hat{\emptyset} \qquad 28.05$$

Both speeds and accelerations are positive.

$$\vec{a}'=\frac{d\vec{v}'}{dt'}=\frac{dt}{dt'}\frac{d}{dt}\left(\frac{\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$\vec{F}' = \frac{\mathrm{m_o}\vec{\mathrm{a}}'}{\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}} = \vec{\mathrm{F}} = \frac{\mathrm{m_o}}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)\frac{\mathrm{d}\vec{\mathrm{v}}}{\mathrm{dt}} + \mathrm{v}\frac{\mathrm{dv}}{\mathrm{dt}}\frac{\vec{\mathrm{v}}}{\mathrm{c}^2}\right] \quad 28.06$$

$$E_k = \int \vec{F}'.d\vec{r} = \int \vec{F}.d\vec{r} = \int -\frac{k}{r^2}\hat{r}.d\vec{r} \quad 28.07$$

$$E_k = \int \frac{m_o\vec{\mathrm{a}}'}{\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}}.d\vec{r} = \int \frac{m_o}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)\frac{d\vec{\mathrm{v}}}{dt} + \mathrm{v}\frac{d\mathrm{v}}{dt}\frac{\vec{\mathrm{v}}}{\mathrm{c}^2}\right].d\vec{r} = \int -\frac{k}{r^2}\hat{r}.d\vec{r}$$

$$E_k = \int \frac{m_o\mathrm{v}'\mathrm{dv}'}{\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}} = \int \frac{m_o\mathrm{vdv}}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} = \int -\frac{k}{r^2}dr \qquad dE_k = \vec{F}.d\vec{r} = \frac{m_o\mathrm{v}'\mathrm{dv}'}{\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}} = \frac{m_o\mathrm{vdv}}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2}dr \quad 28.08$$

$$E_k = m_o\mathrm{c}^2\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}} = \frac{m_o\mathrm{c}^2}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} = \frac{k}{r} + constante \qquad E_R = m_o\mathrm{c}^2\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}} - \frac{k}{r} = m_o\mathrm{c}^2 \quad 28.09$$

$$E_R = \frac{m_o\mathrm{c}^2}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} - \frac{k}{r} = m_o\mathrm{c}^2 \qquad \frac{1}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} = \left(1+\frac{k}{m_o\mathrm{c}^2}\frac{1}{r}\right)^3 = \left(1+A\frac{1}{r}\right)^3 \quad 28.10$$

In this first variant relativistic kinetic energy is greater than inertial energy $\frac{m_o\mathrm{c}^2}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} > m_o\mathrm{c}^2$.This causes Mercury's perihelion to recede. The planet seems heavier due to the movement.

$$\frac{dE_k}{dt} = \vec{F}.\frac{d\vec{r}}{dt} = \frac{m_o\mathrm{v}'\frac{\mathrm{dv}'}{dt'}}{\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}} = \frac{m_o\mathrm{v}\frac{\mathrm{dv}}{dt}}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2}dr = -\frac{k}{r^2}\hat{r}.\frac{d\vec{r}}{dt}$$

$$\frac{dE_k}{dt} = \vec{F}.\vec{\mathrm{v}} = \frac{m_o\vec{\mathrm{v}}'\frac{\mathrm{d}\vec{\mathrm{v}}'}{dt'}}{\left(1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}\right)} = \frac{m_o\vec{\mathrm{v}}\frac{d\vec{\mathrm{v}}}{dt}}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2}dr = -\frac{k}{r^2}\hat{r}.\vec{\mathrm{v}}$$

$$\frac{dE_k}{dt} = \vec{F}.\vec{\mathrm{v}} = \frac{m_o\vec{\mathrm{v}}'.\vec{\mathrm{a}}'}{\left(1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}\right)} = \frac{m_o\vec{\mathrm{v}}.\vec{a}}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2}\hat{r}.\vec{\mathrm{v}} \qquad \vec{F} = \frac{m_o\vec{a}}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2}\hat{r} \quad 28.11$$

$$\vec{F} = \frac{m_o}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}}\left\{\left[\frac{d^2\mathrm{r}}{dt^2} - \mathrm{r}\left(\frac{d\emptyset}{dt}\right)^2\right]\hat{\mathrm{r}} + \left(2\frac{\mathrm{dr}}{dt}\frac{d\emptyset}{dt} + \mathrm{r}\frac{d^2\emptyset}{dt^2}\right)\hat{\emptyset}\right\} = -\frac{k}{r^2}\hat{r}$$

$$\vec{F}_{\hat{\emptyset}} = \left(2\frac{\mathrm{dr}}{dt}\frac{d\emptyset}{dt} + \mathrm{r}\frac{d^2\emptyset}{dt^2}\right)\hat{\emptyset} = zero \qquad \frac{dL}{dt} = \frac{d}{dt}\left(r^2\frac{d\emptyset}{dt}\right) = 2r\frac{\mathrm{dr}}{dt}\frac{d\emptyset}{dt} + \mathrm{r}^2\frac{d^2\emptyset}{dt^2} = zero$$

$$\vec{F}_{\hat{r}} = \frac{m_o}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}}\left[\frac{d^2\mathrm{r}}{dt^2} - \mathrm{r}\left(\frac{d\emptyset}{dt}\right)^2\right]\hat{r} = -\frac{k}{r^2}\hat{r}$$

$$\frac{m_o}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}}\left[\frac{d^2\mathrm{r}}{dt^2} - \mathrm{r}\left(\frac{d\emptyset}{dt}\right)^2\right] = -\frac{k}{r^2}$$

$$\frac{d\emptyset}{dt} = \frac{L}{r^2} \qquad \frac{dr}{dt} = -\mathrm{L}\frac{dw}{d\emptyset} \qquad \frac{d^2r}{dt^2} = \frac{-\mathrm{L}^2}{r^2}\frac{d^2w}{d\emptyset^2}$$

$$\frac{1}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}}\left[\frac{-\mathrm{L}^2}{r^2}\frac{d^2w}{d\emptyset^2} - \mathrm{r}\left(\frac{L}{r^2}\right)^2\right] = -\frac{k}{m_o}\frac{1}{r^2}$$

$$\frac{1}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}}\left[\frac{\mathrm{L}^2}{r^2}\frac{d^2w}{d\emptyset^2} + \mathrm{r}\left(\frac{L}{r^2}\right)^2\right] = \frac{k}{m_o}\frac{1}{r^2}$$

$$\frac{1}{\left(1-\frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}}\left(\frac{d^2w}{d\emptyset^2} + \frac{1}{r}\right) = \frac{k}{m_o\mathrm{L}^2} \qquad A = \frac{k}{m_oc^2} \qquad B = \frac{k}{m_oL^2}$$

$$\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \left(1 + A\frac{1}{r}\right)^3 = 1^3 + 3A\frac{1}{r} + 3A^2\frac{1}{r^2} + A^3\frac{1}{r^3} \cong 1 + 3A\frac{1}{r} \qquad 3A^2\frac{1}{r^2} + A^3\frac{1}{r^3} \cong zero$$

$$\left(1 + 3A\frac{1}{r}\right)\left(\frac{d^2w}{d\phi^2} + \frac{1}{r}\right) = B \qquad 28.12$$

$$\left(1 + 3A\frac{1}{r}\right)\frac{d^2w}{d\phi^2} + \left(1 + 3A\frac{1}{r}\right)\frac{1}{r} - B = zero$$

$$\frac{d^2w}{d\phi^2} + 3A\frac{d^2w}{d\phi^2}\frac{1}{r} + \frac{1}{r} + 3A\frac{1}{r^2} - B = zero$$

$$\frac{d^2w}{d\phi^2} + 3A\frac{d^2w}{d\phi^2}w + \mathrm{w} + 3Aw^2 - B = zero$$

$$\frac{d^2w}{d\phi^2} + \mathrm{w} + 3A\frac{d^2w}{d\phi^2}w + 3Aw^2 - B = zero \qquad 28.13$$

$$w = \frac{1}{r} = \frac{1}{\varepsilon D}[1 + \varepsilon cos(\phi Q)] \qquad \frac{dw}{d\phi} = \frac{-Qsen(\phi Q)}{D} \qquad \frac{d^2w}{d\phi^2} = \frac{-Q^2cos(\phi Q)}{D}$$

$$\frac{-Q^2cos(\phi Q)}{D} + \frac{1}{\varepsilon D}[1 + \varepsilon cos(\phi Q)] + 3A\frac{-Q^2cos(\phi Q)}{D}\frac{1}{\varepsilon D}[1 + \varepsilon cos(\phi Q)] + 3A\left\{\frac{1}{\varepsilon D}[1 + \varepsilon cos(\phi Q)]\right\}^2 - B = zero$$

$$-Q^2\frac{cos(\phi Q)}{D} + \frac{1}{\varepsilon D} + \frac{cos(\phi Q)}{D} - 3AQ^2\frac{1}{\varepsilon D}\frac{cos(\phi Q)}{D} - 3AQ^2\frac{cos^2(\phi Q)}{D^2} + \frac{3A}{\varepsilon^2D^2} + \frac{6A}{\varepsilon D}\frac{cos(\phi Q)}{D} + 3A\frac{cos^2(\phi Q)}{D^2} - B = zero$$

$$(3A - 3AQ^2)\frac{cos^2(\phi Q)}{D^2} + \left(1 - Q^2 - 3AQ^2\frac{1}{\varepsilon D} + \frac{6A}{\varepsilon D}\right)\frac{cos(\phi Q)}{D} + \frac{1}{\varepsilon D} + \frac{3A}{\varepsilon^2D^2} - B = zero$$

$$\left(\frac{3A}{3A} - \frac{3AQ^2}{3A}\right)\frac{cos^2(\phi Q)}{D^2} + \left(\frac{1}{3A} - \frac{Q^2}{3A} - \frac{3AQ^2}{3A}\frac{1}{\varepsilon D} + \frac{6A}{3A\varepsilon D}\right)\frac{cos(\phi Q)}{D} + \frac{1}{3A\varepsilon D} + \frac{3A}{3A\varepsilon^2D^2} - \frac{\varepsilon DB}{\varepsilon D3A} = zero$$

$$\varepsilon DB = \frac{\varepsilon D\mathrm{k}}{\mathrm{m_o}L^2} = \frac{\varepsilon DGM_o\mathrm{m_o}}{\mathrm{m_o GM_o}\varepsilon D} = 1$$

$$(1 - Q^2)\frac{cos^2(\phi Q)}{D^2} + \left(\frac{1}{3A} - \frac{Q^2}{3A} - \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D}\right)\frac{cos(\phi Q)}{D} + \frac{1}{3A\varepsilon D} + \frac{1}{\varepsilon^2D^2} - \frac{1}{\varepsilon D3A} = zero$$

$$(1 - Q^2)\frac{cos^2(\phi Q)}{D^2} + \left(\frac{1}{3A} - \frac{Q^2}{3A} - \frac{Q^2}{\varepsilon D} + \frac{2}{\varepsilon D}\right)\frac{cos(\phi Q)}{D} + \frac{1}{\varepsilon^2D^2} = zero \qquad Q^2 = \frac{1+\frac{12A}{\varepsilon D}}{1+\frac{6A}{\varepsilon D}} \qquad 28.14$$

$$\left[1 - \left(\frac{1+\frac{12A}{\varepsilon D}}{1+\frac{6A}{\varepsilon D}}\right)\right]\frac{cos^2(\phi Q)}{D^2} + \left[\frac{1}{3A} - \frac{1}{3A}\left(\frac{1+\frac{12A}{\varepsilon D}}{1+\frac{6A}{\varepsilon D}}\right) - \frac{1}{\varepsilon D}\left(\frac{1+\frac{12A}{\varepsilon D}}{1+\frac{6A}{\varepsilon D}}\right) + \frac{2}{\varepsilon D}\right]\frac{cos(\phi Q)}{D} + \frac{1}{\varepsilon^2D^2} = zero$$

$$\left[1 + \frac{6A}{\varepsilon D} - 1 - \frac{12A}{\varepsilon D}\right]\frac{cos^2(\phi Q)}{D^2} + \left[\frac{1}{3A}\left(1 + \frac{6A}{\varepsilon D}\right) - \frac{1}{3A}\left(1 + \frac{12A}{\varepsilon D}\right) - \frac{1}{\varepsilon D}\left(1 + \frac{12A}{\varepsilon D}\right) + \frac{2}{\varepsilon D}\left(1 + \frac{6A}{\varepsilon D}\right)\right]\frac{cos(\phi Q)}{D} + \frac{1}{\varepsilon^2D^2}\left(1 + \frac{6A}{\varepsilon D}\right) = zero$$

$$-\frac{6A}{\varepsilon D}\frac{cos^2(\phi Q)}{D^2} + \left(\frac{1}{3A} + \frac{2}{\varepsilon D} - \frac{1}{3A} - \frac{4}{\varepsilon D} - \frac{1}{\varepsilon D} - \frac{12A}{\varepsilon^2D^2} + \frac{2}{\varepsilon D} + \frac{12A}{\varepsilon^2D^2}\right)\frac{cos(\phi Q)}{D} + \frac{1}{\varepsilon^2D^2} + \frac{1}{\varepsilon^2D^2}\frac{6A}{\varepsilon D} = zero$$

$$-\frac{6A}{\varepsilon D}\frac{cos^2(\phi Q)}{D^2} + \left(-\frac{1}{\varepsilon D}\right)\frac{cos(\phi Q)}{D} + \frac{1}{\varepsilon^2D^2} + \frac{1}{\varepsilon^2D^2}\frac{6A}{\varepsilon D} = zero$$

$$6A\frac{cos^2(\phi Q)}{D^2} + \frac{cos(\phi Q)}{D} - \frac{1}{\varepsilon D} - \frac{6A}{\varepsilon^2D^2} = zero$$

$$\frac{cos(\phi Q)}{D} = \frac{-1\pm\sqrt{1-4.6A\left(-\frac{1}{\varepsilon D}-\frac{6A}{\varepsilon^2D^2}\right)}}{2.6A}$$

$$\frac{cos(\phi Q)}{D} = \frac{-1\pm\sqrt{1+\frac{24A}{\varepsilon D}+\frac{144A^2}{\varepsilon^2D^2}}}{2.6A}$$

$$\frac{cos(\phi Q)}{D} = \frac{-1\pm\sqrt{\left(1+\frac{12A}{\varepsilon D}\right)^2}}{12A}$$

$$\frac{cos(\emptyset Q)}{D} = \frac{-1 \pm \left(1+\frac{12A}{\varepsilon D}\right)}{12A}$$

$$\frac{cos(\emptyset Q)}{D} = \frac{-1+1+\frac{12A}{\varepsilon D}}{12A} = \frac{1}{\varepsilon D}$$

$$\frac{cos(\emptyset Q)}{D} = \frac{1}{\varepsilon D} \qquad \varepsilon - \frac{1}{cos(\emptyset Q)} = zero \qquad 28.15$$

For hyperbole eccentricity (ε) is defined as $\varepsilon = \frac{1}{cos(\emptyset Q)}$ where ($\emptyset$) is the angle of the asymptote.

Advance of the Periellium $\quad$ $\mathbf{Q < 1}$

$$\mathrm{dt} = \mathrm{dt}'\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}} \qquad \mathrm{dt}' = \mathrm{dt}\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}} \qquad \mathrm{dt} > dt'$$

$$\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}} = 1$$

$$\mathrm{v} = \frac{\mathrm{v}'}{\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}} \qquad \mathrm{v}' = \frac{\mathrm{v}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} \qquad \mathrm{v}' > \mathrm{v}$$

$$\vec{\mathrm{r}} = \mathrm{r}\hat{\mathrm{r}} \qquad d\vec{\mathrm{r}} = \mathrm{dr}\hat{\mathrm{r}} + \mathrm{r}\mathrm{d}\hat{\mathrm{r}} \qquad \hat{r}.d\vec{\mathrm{r}} = \mathrm{dr}\hat{\mathrm{r}}.\hat{\mathrm{r}} + \mathrm{r}\hat{\mathrm{r}}.\mathrm{d}\hat{\mathrm{r}} = dr$$

$$\vec{\mathrm{v}} = \frac{d\vec{\mathrm{r}}}{dt} = \frac{\mathrm{dr}}{dt}\hat{\mathrm{r}} + \mathrm{r}\frac{d\emptyset}{dt}\hat{\emptyset} \qquad \vec{\mathrm{v}}' = \frac{d\vec{\mathrm{r}}}{dt'} = \frac{\mathrm{dr}}{dt'}\hat{\mathrm{r}} + \mathrm{r}\frac{d\emptyset}{dt'}\hat{\emptyset}$$

$$\vec{\mathrm{v}} = \frac{\vec{\mathrm{v}}'}{\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}} \qquad \vec{\mathrm{v}}' = \frac{\vec{\mathrm{v}}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}}$$

$$\vec{a} = \frac{d\vec{\mathrm{v}}}{dt} = \frac{d^2\vec{\mathrm{r}}}{dt^2} = \frac{d^2(r\hat{r})}{dt^2} = \left[\frac{d^2\mathrm{r}}{dt^2} - \mathrm{r}\left(\frac{d\emptyset}{dt}\right)^2\right]\hat{\mathrm{r}} + \left(2\frac{\mathrm{dr}}{dt}\frac{d\emptyset}{dt} + \mathrm{r}\frac{d^2\emptyset}{dt^2}\right)\hat{\emptyset}$$

$$\vec{a}' = \frac{d\vec{\mathrm{v}}'}{dt'} = \frac{d^2\vec{\mathrm{r}}}{dt'^2} = \frac{d^2(r\hat{r})}{dt'^2} = \left[\frac{d^2\mathrm{r}}{dt'^2} - \mathrm{r}\left(\frac{d\emptyset}{dt'}\right)^2\right]\hat{\mathrm{r}} + \left(2\frac{\mathrm{dr}}{dt'}\frac{d\emptyset}{dt'} + \mathrm{r}\frac{d^2\emptyset}{dt'^2}\right)\hat{\emptyset}$$

$$\vec{a} = \frac{d\vec{\mathrm{v}}}{dt} = \frac{dt'}{dt}\frac{d}{dt'}\left(\frac{\vec{\mathrm{v}}'}{\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}}\right)$$

$$\vec{F} = \frac{m_o\vec{\mathrm{a}}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} = \vec{F}' = \frac{m_o}{\left(1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}\right)\frac{d\vec{\mathrm{v}}'}{dt'} - \mathrm{v}'\frac{d\mathrm{v}'}{dt'}\frac{\vec{\mathrm{v}}'}{\mathrm{c}^2}\right] \qquad 28.16$$

$$E_k = \int \vec{F}.d\vec{r} = \int \vec{F}'.d\vec{r} = \int -\frac{k}{r^2}\hat{r}.d\vec{r}$$

$$E_k = \int \frac{m_o\vec{\mathrm{a}}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}}.d\vec{r} = \int \frac{m_o}{\left(1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}\right)\frac{d\vec{\mathrm{v}}'}{dt'} - \mathrm{v}'\frac{d\mathrm{v}'}{dt'}\frac{\vec{\mathrm{v}}'}{\mathrm{c}^2}\right].d\vec{r} = \int -\frac{k}{r^2}\hat{r}.d\vec{r}$$

$$E_k = \int \frac{m_o\mathrm{v}\mathrm{dv}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} = \int \frac{m_o\mathrm{v}'d\mathrm{v}'}{\left(1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} = \int -\frac{k}{r^2}dr \qquad dE_k = \vec{F}'.d\vec{r} = \frac{m_o\mathrm{v}\mathrm{dv}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} = \frac{m_o\mathrm{v}'d\mathrm{v}'}{\left(1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2}dr \qquad 28.17$$

$$E_k = -m_o\mathrm{c}^2\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}} = -\frac{m_o\mathrm{c}^2}{\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}} = \frac{k}{r} + constante \qquad E_R = -m_o\mathrm{c}^2\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}} - \frac{k}{r} = -m_o\mathrm{c}^2 \qquad 28.18$$

$$E_R = -\frac{m_o\mathrm{c}^2}{\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}} - \frac{k}{r} = -m_o\mathrm{c}^2 \qquad \frac{1}{\left(1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} = \left(1 - \frac{k}{m_o\mathrm{c}^2}\frac{1}{r}\right)^3 = \left(1 - A\frac{1}{r}\right)^3 \qquad 28.19$$

In this second variant relativistic kinetic energy is smaller than inertial energy $\frac{m_o\mathrm{c}^2}{\sqrt{1+\frac{\mathrm{v}'^2}{\mathrm{c}^2}}} < m_o\mathrm{c}^2$.This causes the advance of Mercury's perihelion. The planet really is lighter due to movement.

$$\frac{dE_k}{dt'} = \vec{F}'.\frac{d\vec{r}}{dt'} = \frac{m_o \mathrm{v}\frac{d\mathrm{v}}{dt}}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = \frac{m_o \mathrm{v}'\frac{d\mathrm{v}'}{dt'}}{\left(1+\frac{\mathrm{v}'^2}{c^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2}\frac{dr}{dt'} = -\frac{k}{r^2}\hat{r}.\frac{d\vec{r}}{dt'}$$

$$\frac{dE_k}{dt'} = \vec{F}'.\vec{\mathrm{v}}' = \frac{m_o \vec{\mathrm{v}}\frac{d\vec{\mathrm{v}}}{dt}}{\left(1-\frac{\mathrm{v}^2}{c^2}\right)} = \frac{m_o \vec{\mathrm{v}}'\frac{d\vec{\mathrm{v}}'}{dt}}{\left(1+\frac{\mathrm{v}'^2}{c^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2}\hat{r}.\vec{\mathrm{v}}'$$

$$\frac{dE_k}{dt} = \vec{F}'.\vec{\mathrm{v}}' = \frac{m_o \vec{\mathrm{v}}.\vec{a}}{\left(1-\frac{\mathrm{v}^2}{c^2}\right)} = \frac{m_o \vec{\mathrm{v}}'.\vec{a}'}{\left(1+\frac{\mathrm{v}'^2}{c^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2}\hat{r}.\vec{\mathrm{v}}' \qquad \vec{F}' = \frac{m_o \vec{a}'}{\left(1+\frac{\mathrm{v}'^2}{c^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2}\hat{r} \qquad 28.20$$

$$\vec{F}' = \frac{m_o}{\left(1+\frac{\mathrm{v}'^2}{c^2}\right)^{\frac{3}{2}}}\left\{\left[\frac{d^2\mathrm{r}}{dt'^2} - \mathrm{r}\left(\frac{d\emptyset}{dt'}\right)^2\right]\hat{\mathrm{r}} + \left(2\frac{\mathrm{dr}}{dt'}\frac{d\emptyset}{dt'} + \mathrm{r}\frac{d^2\emptyset}{dt'^2}\right)\hat{\emptyset}\right\} = -\frac{k}{r^2}\hat{r}$$

$$\vec{F}'_{\hat{\emptyset}} = \left(2\frac{\mathrm{dr}}{dt'}\frac{d\emptyset}{dt'} + \mathrm{r}\frac{d^2\emptyset}{dt'^2}\right)\hat{\emptyset} = zero \qquad \frac{dL'}{dt'} = \frac{d}{dt'}\left(r^2\frac{d\emptyset}{dt'}\right) = 2r\frac{\mathrm{dr}}{dt'}\frac{d\emptyset}{dt'} + \mathrm{r}^2\frac{d^2\emptyset}{dt'^2} = zero$$

$$\vec{F}'_{\hat{r}} = \frac{m_o}{\left(1+\frac{\mathrm{v}'^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{d^2\mathrm{r}}{dt'^2} - \mathrm{r}\left(\frac{d\emptyset}{dt'}\right)^2\right]\hat{r} = -\frac{k}{r^2}\hat{r}$$

$$\frac{m_o}{\left(1+\frac{\mathrm{v}'^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{d^2\mathrm{r}}{dt'^2} - \mathrm{r}\left(\frac{d\emptyset}{dt'}\right)^2\right] = -\frac{k}{r^2}$$

$$\frac{d\emptyset}{dt'} = \frac{L'}{r^2} \qquad \frac{dr}{dt'} = -\mathrm{L}'\frac{dw}{d\emptyset} \qquad \frac{d^2r}{dt'^2} = \frac{-\mathrm{L}'^2}{r^2}\frac{d^2w}{d\emptyset^2}$$

$$\frac{1}{\left(1+\frac{\mathrm{v}'^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{-L'^2}{r^2}\frac{d^2w}{d\emptyset^2} - \mathrm{r}\left(\frac{L'}{r^2}\right)^2\right] = -\frac{k}{m_o}\frac{1}{r^2}$$

$$\frac{1}{\left(1+\frac{\mathrm{v}'^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{L'^2}{r^2}\frac{d^2w}{d\emptyset^2} + \mathrm{r}\left(\frac{L'}{r^2}\right)^2\right] = \frac{k}{m_o}\frac{1}{r^2}$$

$$\frac{1}{\left(1+\frac{\mathrm{v}'^2}{c^2}\right)^{\frac{3}{2}}}\left(\frac{d^2w}{d\emptyset^2} + \frac{1}{r}\right) = \frac{k}{m_o L'^2} \qquad A = \frac{k}{m_o c^2} \qquad B = \frac{k}{m_o L'^2}$$

$$\frac{1}{\left(1+\frac{\mathrm{v}'^2}{c^2}\right)^{\frac{3}{2}}} = \left(1 - A\frac{1}{r}\right)^3 = 1 - 3A\frac{1}{r} + 3A^2\frac{1}{r^2} - A^3\frac{1}{r^3} \cong 1 - 3A\frac{1}{r} \qquad 3A^2\frac{1}{r^2} - A^3\frac{1}{r^3} \cong zero$$

$$\left(1 - 3A\frac{1}{r}\right)\left(\frac{d^2w}{d\emptyset^2} + \frac{1}{r}\right) = B \qquad 28.21$$

$$\left(1 - 3A\frac{1}{r}\right)\frac{d^2w}{d\emptyset^2} + \left(1 - 3A\frac{1}{r}\right)\frac{1}{r} - B = zero$$

$$\frac{d^2w}{d\emptyset^2} - 3A\frac{d^2w}{d\emptyset^2}\frac{1}{r} + \frac{1}{r} - 3A\frac{1}{r^2} - B = zero$$

$$\frac{d^2w}{d\emptyset^2} - 3A\frac{d^2w}{d\emptyset^2}w + \mathrm{w} - 3Aw^2 - B = zero$$

$$\frac{d^2w}{d\emptyset^2} + \mathrm{w} - 3A\frac{d^2w}{d\emptyset^2}w - 3Aw^2 - B = zero \qquad 28.22$$

$$w = \frac{1}{r} = \frac{1}{\varepsilon D}[1 + \varepsilon cos(\emptyset Q)] \qquad \frac{dw}{d\emptyset} = \frac{-Qsen(\emptyset Q)}{D} \qquad \frac{d^2w}{d\emptyset^2} = \frac{-Q^2cos(\emptyset Q)}{D}$$

$$\frac{-Q^2cos(\emptyset Q)}{D} + \frac{1}{\varepsilon D}[1 + \varepsilon cos(\emptyset Q)] - 3A\frac{-Q^2cos(\emptyset Q)}{D}\frac{1}{\varepsilon D}[1 + \varepsilon cos(\emptyset Q)] - 3A\left\{\frac{1}{\varepsilon D}[1 + \varepsilon cos(\emptyset Q)]\right\}^2 - B = zero$$

$$-Q^2\frac{cos(\emptyset Q)}{D}+\frac{1}{\varepsilon D}+\frac{cos(\emptyset Q)}{D}+3AQ^2\frac{1}{\varepsilon D}\frac{cos(\emptyset Q)}{D}+3AQ^2\frac{cos^2(\emptyset Q)}{D^2}-\left[\frac{3A}{\varepsilon^2 D^2}+\frac{6A}{\varepsilon D}\frac{cos(\emptyset Q)}{D}+3A\frac{cos^2(\emptyset Q)}{D^2}\right]-B=zero$$

$$-Q^2\frac{cos(\emptyset Q)}{D}+\frac{1}{\varepsilon D}+\frac{cos(\emptyset Q)}{D}+3AQ^2\frac{1}{\varepsilon D}\frac{cos(\emptyset Q)}{D}+3AQ^2\frac{cos^2(\emptyset Q)}{D^2}-\frac{3A}{\varepsilon^2 D^2}-\frac{6A}{\varepsilon D}\frac{cos(\emptyset Q)}{D}-3A\frac{cos^2(\emptyset Q)}{D^2}-B=zero$$

$$(3AQ^2-3A)\frac{cos^2(\emptyset Q)}{D^2}+\left(1-Q^2+3AQ^2\frac{1}{\varepsilon D}-\frac{6A}{\varepsilon D}\right)\frac{cos(\emptyset Q)}{D}+\frac{1}{\varepsilon D}-\frac{3A}{\varepsilon^2 D^2}-B=zero$$

$$\left(\frac{3AQ^2}{3A}-\frac{3A}{3A}\right)\frac{cos^2(\emptyset Q)}{D^2}+\left(\frac{1}{3A}-\frac{Q^2}{3A}+\frac{3AQ^2}{3A}\frac{1}{\varepsilon D}-\frac{6A}{3A\varepsilon D}\right)\frac{cos(\emptyset Q)}{D}+\frac{1}{3A\varepsilon D}-\frac{3A}{3A\varepsilon^2 D^2}-\frac{B}{3A}=zero$$

$$(Q^2-1)\frac{cos^2(\emptyset Q)}{D^2}+\left(\frac{1}{3A}-\frac{Q^2}{3A}+Q^2\frac{1}{\varepsilon D}-\frac{2}{\varepsilon D}\right)\frac{cos(\emptyset Q)}{D}+\frac{1}{3A\varepsilon D}-\frac{1}{\varepsilon^2 D^2}-\frac{\varepsilon DB}{\varepsilon D3A}=zero$$

$$\varepsilon DB=\frac{\varepsilon Dk}{m_o L'^2}=\frac{\varepsilon DGM_o \mathrm{m_o}}{\mathrm{m_o GM_o}\varepsilon D}=1$$

$$(1-Q^2)\frac{cos^2(\emptyset Q)}{D^2}+\left(-\frac{1}{3A}+\frac{Q^2}{3A}-Q^2\frac{1}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{cos(\emptyset Q)}{D}-\frac{1}{3A\varepsilon D}+\frac{1}{\varepsilon^2 D^2}+\frac{1}{\varepsilon D3A}=zero$$

$$(1-Q^2)\frac{cos^2(\emptyset Q)}{D^2}+\left(-\frac{1}{3A}+\frac{Q^2}{3A}-Q^2\frac{1}{\varepsilon D}+\frac{2}{\varepsilon D}\right)\frac{cos(\emptyset Q)}{D}+\frac{1}{\varepsilon^2 D^2}=zero \quad Q^2=\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}$$ 28.23

$$\left[1-\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)\right]\frac{cos^2(\emptyset Q)}{D^2}+\left[-\frac{1}{3A}+\frac{1}{3A}\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)-\frac{1}{\varepsilon D}\left(\frac{1-\frac{12A}{\varepsilon D}}{1-\frac{6A}{\varepsilon D}}\right)+\frac{2}{\varepsilon D}\right]\frac{cos(\emptyset Q)}{D}+\frac{1}{\varepsilon^2 D^2}=zero$$

$$\left(1-\frac{6A}{\varepsilon D}-1+\frac{12A}{\varepsilon D}\right)\frac{cos^2(\emptyset Q)}{D^2}+\left[-\frac{1}{3A}\left(1-\frac{6A}{\varepsilon D}\right)+\frac{1}{3A}\left(1-\frac{12A}{\varepsilon D}\right)-\frac{1}{\varepsilon D}\left(1-\frac{12A}{\varepsilon D}\right)+\frac{2}{\varepsilon D}\left(1-\frac{6A}{\varepsilon D}\right)\right]\frac{cos(\emptyset Q)}{D}+\frac{1}{\varepsilon^2 D^2}\left(1-\frac{6A}{\varepsilon D}\right)=zero$$

$$\frac{6A}{\varepsilon D}\frac{cos^2(\emptyset Q)}{D^2}+\left(-\frac{1}{3A}+\frac{2}{\varepsilon D}+\frac{1}{3A}-\frac{4}{\varepsilon D}-\frac{1}{\varepsilon D}+\frac{12A}{\varepsilon^2 D^2}+\frac{2}{\varepsilon D}-\frac{12A}{\varepsilon^2 D^2}\right)\frac{cos(\emptyset Q)}{D}+\frac{1}{\varepsilon^2 D^2}-\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}=zero$$

$$\frac{6A}{\varepsilon D}\frac{cos^2(\emptyset Q)}{D^2}+\left(-\frac{1}{\varepsilon D}\right)\frac{cos(\emptyset Q)}{D}+\frac{1}{\varepsilon^2 D^2}-\frac{1}{\varepsilon^2 D^2}\frac{6A}{\varepsilon D}=zero$$

$$6A\frac{cos^2(\emptyset Q)}{D^2}-\frac{cos(\emptyset Q)}{D}+\frac{1}{\varepsilon D}-\frac{6A}{\varepsilon^2 D^2}=zero$$

$$\frac{cos(\emptyset Q)}{D}=\frac{1\pm\sqrt{1-4.6A\left(\frac{1}{\varepsilon D}-\frac{6A}{\varepsilon^2 D^2}\right)}}{2.6A}$$

$$\frac{cos(\emptyset Q)}{D}=\frac{1\pm\sqrt{1-\frac{24A}{\varepsilon D}+\frac{144A^2}{\varepsilon^2 D^2}}}{2.6A}$$

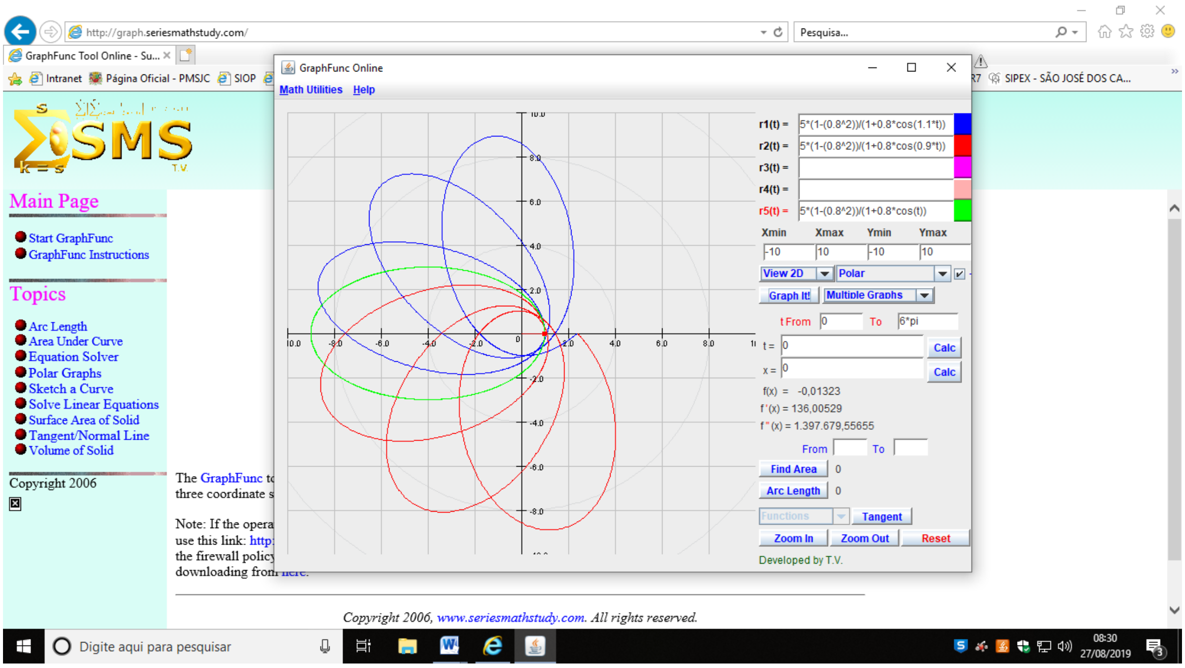
$$\frac{cos(\emptyset Q)}{D}=\frac{1\pm\sqrt{\left(1-\frac{12A}{\varepsilon D}\right)^2}}{12A}$$

$$\frac{cos(\emptyset Q)}{D}=\frac{1\pm\left(1-\frac{12A}{\varepsilon D}\right)}{12A}$$

$$\frac{cos(\emptyset Q)}{D}=\frac{1-1+\frac{12A}{\varepsilon D}}{12A}=\frac{1}{\varepsilon D}$$

$$\frac{cos(\emptyset Q)}{D}=\frac{1}{\varepsilon D} \qquad \varepsilon-\frac{1}{cos(\emptyset Q)}=zero$$ 28.24

For hyperbole eccentricity (ε) is defined as $\varepsilon=\frac{1}{cos(\emptyset Q)}$ where ($\emptyset$) is the angle of the asymptote.



The movements of the ellipses will focus F '(left) on the origin of the frame.

All ellipses are described by the equation $r = r(t) = \frac{\varepsilon D}{1+\varepsilon cos(tQ)} = \frac{a(1-\varepsilon^2)}{1+\varepsilon cos(tQ)} = \frac{5(1-0.8^2)}{1+0.8cos(tQ)}$ In these the angle vector radius (tQ), indicates the position of the planet Mercury in all ellipses, the movement of Mercury in the ellipses is counterclockwise, with the value of Q being the cause of perihelion advancement or retraction.

The first ellipse in blue represents retrogression of the perihelion, where we have Q = 1.1.

The second red ellipse represents the advancement of the perihelion, in this we have Q = 0.9. In this ellipse the perihelion and aphelion advance in the trigonometric sense, that is, counterclockwise which is the same direction as the planet's movement in the ellipse.

The fifth ellipse in green represents a stationary ellipse Q = 1.

29 Yukawa Potential Energy

Newton's gravitational potential energy E_{pN}

$\vec{F} = -\frac{k}{r^2}\hat{r}$ $F = |\vec{F}| = \sqrt{\vec{F}.\vec{F}} = \sqrt{\left(-\frac{k}{r^2}\hat{r}\right)\left(-\frac{k}{r^2}\hat{r}\right)} = \sqrt{\left(\frac{k}{r^2}\right)^2 \hat{r}\hat{r}} = \sqrt{\left(\frac{k}{r^2}\right)^2} = \frac{k}{r^2}$

$\vec{F} = -F\hat{r}$ $E_{pN} = -\frac{k}{r}$ $F = \frac{dE_p}{dr} = \frac{k}{r^2}$ k>zero

$\vec{F} = -\frac{dE_p}{dr}\hat{r} = -\frac{k}{r^2}\hat{r}$

Yukawa potential energy E_{pY}

$E_{pY} = -k\frac{e^{-ar}}{r} = -kr^{-1}e^{-ar}$ k>zero $a \geq zero$ 29.01

Potential Core Energy E_N

Breaking apart $E_{pY} = -k\frac{e^{-ar}}{r}$ we get:

$E_N = -k\frac{e^{-ar}}{r} = \left(-\frac{k}{r}\right)\left(\frac{1}{e^{ar}}\right) = E_{pN}C_Y$ $C_Y = \frac{1}{e^{ar}}$ k>zero $a \geq zero$

$a = zero \to C_Y = 1 \to E_N = E_{pN}$ $r = \infty \to E_N = zero$

$$\frac{dE_p}{dr} = \frac{d}{dr}(-kr^{-1}e^{-ar}) = -k\left\{\left[(-1)r^{-1-1}\frac{dr}{dr}\right]e^{-ar} + (r^{-1})e^{-ar}\left(-a\frac{dr}{dr}\right)\right\}$$

$$\frac{dE_p}{dr} = \frac{d}{dr}(-kr^{-1}e^{-ar}) = -k(-r^{-2}e^{-ar} - ar^{-1}e^{-ar}) = k\frac{e^{-ar}}{r^2} + ak\frac{e^{-ar}}{r}$$

$$\frac{dE_p}{dr} = \frac{d}{dr}\left(-k\frac{e^{-ar}}{r}\right) = \left(k\frac{e^{-ar}}{r^2} + ak\frac{e^{-ar}}{r}\right)$$

$$E_{pY} = \int dE_p = \int d\left(-k\frac{e^{-ar}}{r}\right) = \int\left(k\frac{e^{-ar}}{r^2} + ak\frac{e^{-ar}}{r}\right)dr = -k\frac{e^{-ar}}{r} + constante$$

$\vec{F} = -\frac{dE_p}{dr}\hat{r} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$ Attractive force

$\vec{F} = m\vec{a} = \frac{m_o\vec{a}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt}$ 21.51

$\vec{F}' = m\vec{a}' = \frac{m_o\vec{a}'}{\sqrt{1+\frac{v'^2}{c^2}}} = \frac{m_o}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{d\vec{v}'}{dt'}$ 21.15

First variant.

$\vec{F} = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$ 21.51

$$E_k = \int \vec{F}.d\vec{r} = \int \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{v}}{dt}.d\vec{r} = \int -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}.d\vec{r}$$

$$E_k = \int \vec{F}.d\vec{r} = \int \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}d\vec{v}.\frac{d\vec{r}}{dt} = \int -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)dr$$

$$E_k = \int \vec{F}.d\vec{r} = \int \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}d\vec{v}.\vec{v} = -\int\left(k\frac{e^{-ar}}{r^2} + ak\frac{e^{-ar}}{r}\right)dr$$

$E_k = \int \vec{F}.d\vec{r} = \int \frac{m_o v dv}{\sqrt{1-\frac{v^2}{c^2}}} = -\int\left(k\frac{e^{-ar}}{r^2} + ak\frac{e^{-ar}}{r}\right)dr$ $dE_k = \vec{F}.d\vec{r} = \frac{m_o v dv}{\sqrt{1-\frac{v^2}{c^2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)dr$

$$E_k = -m_o c^2 \sqrt{1-\frac{v^2}{c^2}} = -\left(-k\frac{e^{-ar}}{r}\right) + constante$$

$$E_k = -m_o c^2 \sqrt{1-\frac{v^2}{c^2}} = k\frac{e^{-ar}}{r} + constante$$

$$E_k = -m_o c^2 \sqrt{1-\frac{v^2}{c^2}} = k\frac{e^{-ar}}{r} - m_o c^2$$

$$E_R = -m_o c^2 \sqrt{1-\frac{v^2}{c^2}} - k\frac{e^{-ar}}{r} = -m_o c^2$$

$$E_R = -m_o c^2 \sqrt{1-\frac{(zero)^2}{c^2}} - k\frac{e^{-a\infty}}{\infty} = -m_o c^2$$

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{m_o c^2}{m_o c^2} - \frac{k}{m_o c^2}\frac{e^{-ar}}{r}$$

$$A = \frac{k}{m_o c^2}$$

$$\sqrt{1-\frac{v^2}{c^2}} = 1 - A\frac{e^{-ar}}{r}$$

$$dE_k = \vec{F}.d\vec{r} = \frac{m_o v dv}{\sqrt{1-\frac{v^2}{c^2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)dr$$

$$\frac{dE_k}{dt} = \vec{F}.\frac{d\vec{r}}{dt} = \frac{m_o v\frac{dv}{dt}}{\sqrt{1-\frac{v^2}{c^2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}\frac{d\vec{r}}{dt}$$

$$\frac{dE_k}{dt} = \vec{F}.\vec{v} = \frac{m_o \vec{v}\frac{d\vec{v}}{dt}}{\sqrt{1-\frac{v^2}{c^2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}\vec{v}$$

$$\vec{F} = \frac{m_o \vec{a}}{\sqrt{1-\frac{v^2}{c^2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$$

$$\vec{F} = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}\left\{\left[\frac{d^2r}{dt^2} - r\left(\frac{d\emptyset}{dt}\right)^2\right]\hat{r} + \left(2\frac{dr}{dt}\frac{d\emptyset}{dt} + r\frac{d^2\emptyset}{dt^2}\right)\hat{\emptyset}\right\} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$$

$$\vec{F}_{\hat{\emptyset}} = \left(2\frac{dr}{dt}\frac{d\emptyset}{dt} + r\frac{d^2\emptyset}{dt^2}\right)\hat{\emptyset} = zero$$

$$\frac{dL}{dt} = \frac{d}{dt}\left(r^2\frac{d\emptyset}{dt}\right) = 2r\frac{dr}{dt}\frac{d\emptyset}{dt} + r^2\frac{d^2\emptyset}{dt^2} = zero$$

$$\vec{F}_{\hat{r}} = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}\left[\frac{d^2r}{dt^2} - r\left(\frac{d\emptyset}{dt}\right)^2\right]\hat{r} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$$

$$\left[\frac{d^2r}{dt^2} - r\left(\frac{d\emptyset}{dt}\right)^2\right] = -\frac{k}{m_o}\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\sqrt{1-\frac{v^2}{c^2}}$$

$$\left[\frac{d^2r}{dt^2} - r\left(\frac{d\emptyset}{dt}\right)^2\right] = -\frac{k}{m_o}\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\sqrt{1-\frac{v^2}{c^2}}$$

$$\left[\frac{d^2r}{dt^2} - r\left(\frac{d\emptyset}{dt}\right)^2\right] = -\frac{k}{m_o}\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\left(1 - A\frac{e^{-ar}}{r}\right)$$

$$\frac{d\emptyset}{dt} = \frac{L}{r^2} \qquad \frac{dr}{dt} = -L\frac{dw}{d\emptyset} \qquad \frac{d^2r}{dt^2} = \frac{-L^2}{r^2}\frac{d^2w}{d\emptyset^2}$$

$$\left[\frac{-L^2}{r^2}\frac{d^2w}{d\emptyset^2} - r\left(\frac{L}{r^2}\right)^2\right] = -\frac{k}{m_o}\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\left(1 - A\frac{e^{-ar}}{r}\right)$$

$$\frac{d^2w}{d\emptyset^2} + \frac{1}{r} = \frac{k}{m_o L^2}r^2\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\left(1 - A\frac{e^{-ar}}{r}\right) \qquad B = \frac{k}{m_o L^2}$$

$$\frac{d^2w}{d\emptyset^2} + \frac{1}{r} = Br^2\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\left(1 - A\frac{e^{-ar}}{r}\right) \qquad B = \frac{k}{m_o L^2}$$

$$\frac{d^2w}{d\emptyset^2}+\frac{1}{r}=\mathrm{Be}^{-\mathrm{ar}}\left(1-\mathrm{A}\frac{\mathrm{e}^{-\mathrm{ar}}}{\mathrm{r}}\right)+\mathrm{aBre}^{-\mathrm{ar}}\left(1-\mathrm{A}\frac{\mathrm{e}^{-\mathrm{ar}}}{\mathrm{r}}\right)$$

$$\frac{d^2w}{d\emptyset^2}+\frac{1}{r}=\mathrm{Be}^{-\mathrm{ar}}-\mathrm{ABe}^{-\mathrm{ar}}\frac{\mathrm{e}^{-\mathrm{ar}}}{\mathrm{r}}+\mathrm{aBre}^{-\mathrm{ar}}-\mathrm{aABre}^{-\mathrm{ar}}\frac{\mathrm{e}^{-\mathrm{ar}}}{\mathrm{r}}$$

$$\frac{d^2w}{d\emptyset^2}+\frac{1}{r}=\mathrm{Be}^{-\mathrm{ar}}-\mathrm{AB}\frac{\mathrm{e}^{-2\mathrm{ar}}}{\mathrm{r}}+\mathrm{aBre}^{-\mathrm{ar}}-\mathrm{aABe}^{-2\mathrm{ar}} \qquad \mathrm{w}=\frac{1}{r} \qquad r=\mathrm{w}^{-1}$$

$$\frac{d^2w}{d\emptyset^2}+\mathrm{w}=\mathrm{Be}^{-\mathrm{aw}^{-1}}-\mathrm{ABwe}^{-2\mathrm{aw}^{-1}}+\mathrm{aBw}^{-1}\mathrm{e}^{-\mathrm{aw}^{-1}}-\mathrm{aABe}^{-2\mathrm{aw}^{-1}}$$

$$\frac{d^2w}{d\emptyset^2}+\mathrm{w}=\mathrm{Be}^{-\mathrm{aw}^{-1}}+\mathrm{aBw}^{-1}\mathrm{e}^{-\mathrm{aw}^{-1}}-\mathrm{ABwe}^{-2\mathrm{aw}^{-1}}-\mathrm{aABe}^{-2\mathrm{aw}^{-1}}$$

$$\frac{d^2w}{d\emptyset^2}+\mathrm{w}=(1+\mathrm{aw}^{-1})\mathrm{Be}^{-\mathrm{aw}^{-1}}-(\mathrm{w}+\mathrm{a})\mathrm{ABe}^{-2\mathrm{aw}^{-1}}$$

$$\frac{d^2w}{d\emptyset^2}+\mathrm{w}=\left[(1+\mathrm{aw}^{-1})-(\mathrm{w}+\mathrm{a})\mathrm{Ae}^{-\mathrm{aw}^{-1}}\right]\mathrm{Be}^{-\mathrm{aw}^{-1}}$$

$$\mathrm{r}=\frac{1}{Q\cos(\emptyset)} \qquad \mathrm{w}=\frac{1}{r}=Q\cos(\emptyset) \qquad \frac{\mathrm{dw}}{d\emptyset}=-Q\mathrm{sen}(\emptyset) \qquad \frac{\mathrm{d}^2\mathrm{w}}{d\emptyset^2}=-Q\cos(\emptyset)$$

$$-Q\cos(\emptyset)+Q\cos(\emptyset)=\left[(1+\mathrm{aw}^{-1})-(\mathrm{w}+\mathrm{a})\mathrm{Ae}^{-\mathrm{aw}^{-1}}\right]\mathrm{Be}^{-\mathrm{aw}^{-1}}$$

$$\mathrm{zero}=\left[(1+\mathrm{aw}^{-1})-(\mathrm{w}+\mathrm{a})\mathrm{Ae}^{-\mathrm{aw}^{-1}}\right]\mathrm{Be}^{-\mathrm{aw}^{-1}}$$

$$(1+\mathrm{aw}^{-1})-(\mathrm{w}+\mathrm{a})\mathrm{Ae}^{-\mathrm{aw}^{-1}}=zero \qquad \mathrm{w}=\frac{1}{r}=Q\cos(\emptyset) \qquad r=\mathrm{w}^{-1}=\frac{1}{Q\cos(\emptyset)}$$

$$\left[1+\frac{\mathrm{a}}{Q\cos(\emptyset)}\right]-[Q\cos(\emptyset)+\mathrm{a}]\mathrm{Ae}^{-\mathrm{aw}^{-1}}=zero$$

$$Q\cos(\emptyset)\left(1+\frac{\mathrm{a}}{Q\cos(\emptyset)}\right)-Q\cos(\emptyset)[Q\cos(\emptyset)+\mathrm{a}]\mathrm{Ae}^{-\mathrm{aw}^{-1}}=zero$$

$$Q\cos(\emptyset)+\mathrm{a}-Q^2\cos^2(\emptyset)\mathrm{Ae}^{-\mathrm{aw}^{-1}}-Q\cos(\emptyset)\mathrm{aAe}^{-\mathrm{aw}^{-1}}=zero$$

$$-Q\cos(\emptyset)-\mathrm{a}+Q^2\cos^2(\emptyset)\mathrm{Ae}^{-\mathrm{aw}^{-1}}+Q\cos(\emptyset)\mathrm{aAe}^{-\mathrm{aw}^{-1}}=zero$$

$$Q^2\cos^2(\emptyset)\mathrm{Ae}^{-\mathrm{aw}^{-1}}-Q\cos(\emptyset)+Q\cos(\emptyset)\mathrm{aAe}^{-\mathrm{aw}^{-1}}-a=zero$$

$$Q^2\cos^2(\emptyset)\mathrm{Ae}^{-\mathrm{aw}^{-1}}-Q\cos(\emptyset)\left(1-\mathrm{aAe}^{-\mathrm{aw}^{-1}}\right)-a=zero$$

$$Q^2\cos^2(\emptyset)\mathrm{Ae}^{-\mathrm{aw}^{-1}}-Q\cos(\emptyset)\left(1-\mathrm{aAe}^{-\mathrm{aw}^{-1}}\right)-a=zero$$

$$Q\cos(\emptyset)=\frac{1-\mathrm{aAe}^{-\mathrm{aw}^{-1}}\pm\sqrt{\left(1-\mathrm{aAe}^{-\mathrm{aw}^{-1}}\right)^2-4\mathrm{Ae}^{-\mathrm{aw}^{-1}}(-a)}}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

$$Q\cos(\emptyset)=\frac{1-\mathrm{aAe}^{-\mathrm{aw}^{-1}}\pm\sqrt{1-2\mathrm{aAe}^{-\mathrm{aw}^{-1}}+\mathrm{a}^2\mathrm{A}^2\mathrm{e}^{-2\mathrm{aw}^{-1}}+4a\mathrm{Ae}^{-\mathrm{aw}^{-1}}}}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

$$Q\cos(\emptyset)=\frac{1-\mathrm{aAe}^{-\mathrm{aw}^{-1}}\pm\sqrt{1+2\mathrm{aAe}^{-\mathrm{aw}^{-1}}+\mathrm{a}^2\mathrm{A}^2\mathrm{e}^{-2\mathrm{aw}^{-1}}}}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

$$Q\cos(\emptyset)=\frac{1-\mathrm{aAe}^{-\mathrm{aw}^{-1}}\pm\sqrt{\left(1+\mathrm{aAe}^{-\mathrm{aw}^{-1}}\right)^2}}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

$$Q\cos(\emptyset)=\frac{1-\mathrm{aAe}^{-\mathrm{aw}^{-1}}\pm\left(1+\mathrm{aAe}^{-\mathrm{aw}^{-1}}\right)}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

$$Q\cos(\emptyset) = \frac{1 - \mathrm{aAe^{-aw^{-1}}} - 1 - \mathrm{aAe^{-aw^{-1}}}}{\mathrm{2Ae^{-aw^{-1}}}}$$

$$Q\cos(\emptyset) = \frac{-\mathrm{aAe^{-aw^{-1}}} - \mathrm{aAe^{-aw^{-1}}}}{\mathrm{2Ae^{-aw^{-1}}}}$$

$\mathrm{w} = \frac{1}{r} = Q\cos(\emptyset) = \frac{-\mathrm{a}-\mathrm{a}}{2} = \frac{-2\mathrm{a}}{2} = -a$ $\qquad r = \frac{-1}{a}$ $\qquad \mathrm{E}_p = \text{constante}$

$$\frac{\mathrm{dE_p}}{\mathrm{dr}} = \frac{\mathrm{d}}{\mathrm{dr}}\left(-\mathrm{k}\frac{\mathrm{e^{-ar}}}{r}\right) = \left(k\frac{\mathrm{e^{-ar}}}{\mathrm{r^2}} + \mathrm{ak}\frac{\mathrm{e^{-ar}}}{\mathrm{r}}\right) = zero$$

$k\frac{\mathrm{e^{-ar}}}{\mathrm{r^2}} + \mathrm{ak}\frac{\mathrm{e^{-ar}}}{\mathrm{r}} = zero$ $\qquad \frac{1}{\mathrm{r}} = -\mathrm{a}$ $\qquad r = \frac{-1}{a}$ $\qquad \mathrm{E}_p = \text{constante}$

$$\mathrm{E_{pY}} = -\mathrm{k}\frac{\mathrm{e^{-ar}}}{r} = -\mathrm{kr^{-1}e^{-ar}} = -\mathrm{k}(-a)\mathrm{e}^{-\mathrm{a}\frac{-1}{a}} = \mathrm{ake}$$

$$Q\cos(\emptyset) = \frac{1 - \mathrm{aAe^{-aw^{-1}}} \pm \left(1 + \mathrm{aAe^{-aw^{-1}}}\right)}{\mathrm{2Ae^{-aw^{-1}}}}$$

$$Q\cos(\emptyset) = \frac{1 - \mathrm{aAe^{-aw^{-1}}} + 1 + \mathrm{aAe^{-aw^{-1}}}}{\mathrm{2Ae^{-aw^{-1}}}}$$

$$\mathrm{w} = \frac{1}{r} = Q\cos(\emptyset) = \frac{2}{\mathrm{2Ae^{-aw^{-1}}}} = \frac{1}{\mathrm{Ae^{-aw^{-1}}}}$$

Second variant.

$$\vec{\mathrm{F}}' = \frac{\mathrm{m_o}}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}}\frac{\mathrm{d}\vec{\mathrm{v}}'}{\mathrm{dt'}} = -\mathrm{k}\left(\frac{\mathrm{e^{-ar}}}{\mathrm{r^2}} + \mathrm{a}\frac{\mathrm{e^{-ar}}}{\mathrm{r}}\right)\hat{\mathrm{r}} \qquad 21.15$$

$$\mathrm{E_k} = \int \vec{\mathrm{F}}'.\mathrm{d}\vec{\mathrm{r}} = \int \frac{\mathrm{m_o}}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}}\frac{\mathrm{d}\vec{\mathrm{v}}'}{\mathrm{dt'}}.\mathrm{d}\vec{\mathrm{r}} = \int -\mathrm{k}\left(\frac{\mathrm{e^{-ar}}}{\mathrm{r^2}} + \mathrm{a}\frac{\mathrm{e^{-ar}}}{\mathrm{r}}\right)\hat{\mathrm{r}}\,.\,\mathrm{d}\vec{\mathrm{r}}$$

$$\mathrm{E_k} = \int \vec{\mathrm{F}}'.\mathrm{d}\vec{\mathrm{r}} = \int \frac{\mathrm{m_o}}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}}\mathrm{d}\vec{\mathrm{v}}'\frac{\mathrm{d}\vec{\mathrm{r}}}{\mathrm{dt'}} = \int -\mathrm{k}\left(\frac{\mathrm{e^{-ar}}}{\mathrm{r^2}} + \mathrm{a}\frac{\mathrm{e^{-ar}}}{\mathrm{r}}\right)\mathrm{dr}$$

$$\mathrm{E_k} = \int \vec{\mathrm{F}}'.\mathrm{d}\vec{\mathrm{r}} = \int \frac{\mathrm{m_o}}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}}\mathrm{d}\vec{\mathrm{v}}'.\vec{\mathrm{v}'} = -\int\left(k\frac{\mathrm{e^{-ar}}}{\mathrm{r^2}} + \mathrm{ak}\frac{\mathrm{e^{-ar}}}{\mathrm{r}}\right)\mathrm{dr}$$

$$\mathrm{E_k} = \int \vec{\mathrm{F}}'.\mathrm{d}\vec{\mathrm{r}} = \int \frac{\mathrm{m_o v'dv'}}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}} = -\int\left(k\frac{\mathrm{e^{-ar}}}{\mathrm{r^2}} + \mathrm{ak}\frac{\mathrm{e^{-ar}}}{\mathrm{r}}\right)\mathrm{dr} \qquad \mathrm{dE_k} = \vec{\mathrm{F}}'.\mathrm{d}\vec{\mathrm{r}} = \frac{\mathrm{m_o v'dv'}}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}} = -\mathrm{k}\left(\frac{\mathrm{e^{-ar}}}{\mathrm{r^2}} + \mathrm{a}\frac{\mathrm{e^{-ar}}}{\mathrm{r}}\right)\mathrm{dr}$$

$$\mathrm{E_k} = \mathrm{m_o c^2}\sqrt{1 + \frac{\mathrm{v'^2}}{\mathrm{c^2}}} = -\left(-k\frac{\mathrm{e^{-ar}}}{r}\right) + constante$$

$$\mathrm{E_k} = \mathrm{m_o c^2}\sqrt{1 + \frac{\mathrm{v'^2}}{\mathrm{c^2}}} = \mathrm{k}\frac{\mathrm{e^{-ar}}}{\mathrm{r}} + constante \qquad \mathrm{E_k} = \mathrm{m_o c^2}\sqrt{1 + \frac{\mathrm{v'^2}}{\mathrm{c^2}}} = \mathrm{k}\frac{\mathrm{e^{-ar}}}{\mathrm{r}} + \mathrm{m_o c^2}$$

$$\mathrm{E_R} = \mathrm{m_o c^2}\sqrt{1 + \frac{\mathrm{v'^2}}{\mathrm{c^2}}} - \mathrm{k}\frac{\mathrm{e^{-ar}}}{\mathrm{r}} = \mathrm{m_o c^2} \qquad \mathrm{E_R} = \mathrm{m_o c^2}\sqrt{1 + \frac{(zero)^2}{\mathrm{c^2}}} - \mathrm{k}\frac{\mathrm{e^{-a\infty}}}{\infty} = \mathrm{m_o c^2}$$

$$\sqrt{1 + \frac{\mathrm{v'^2}}{\mathrm{c^2}}} = \frac{\mathrm{m_o c^2}}{\mathrm{m_o c^2}} + \frac{\mathrm{k}}{\mathrm{m_o c^2}}\frac{\mathrm{e^{-ar}}}{\mathrm{r}} \qquad A = \frac{\mathrm{k}}{\mathrm{m_o c^2}}$$

$$\sqrt{1 + \frac{\mathrm{v'^2}}{\mathrm{c^2}}} = 1 + \mathrm{A}\frac{\mathrm{e^{-ar}}}{\mathrm{r}}$$

$$dE_k = \vec{F}'.d\vec{r} = \frac{m_o v' dv'}{\sqrt{1+\frac{v'^2}{c^2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)dr$$

$$\frac{dE_k}{dt'} = \vec{F'}.\frac{d\vec{r}}{dt'} = \frac{m_o v' \frac{dv'}{dt'}}{\sqrt{1+\frac{v'^2}{c^2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}\,\frac{d\vec{r}}{dt'}$$

$$\frac{dE_k}{dt'} = \vec{F}'.\vec{v}' = \frac{m_o \vec{v}' \frac{d\vec{v}'}{dt'}}{\sqrt{1+\frac{v'^2}{c^2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}\vec{v}'$$

$$\vec{F}' = \frac{m_o \vec{a}'}{\sqrt{1+\frac{v'^2}{c^2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$$

$$\vec{F}' = \frac{m_o}{\sqrt{1+\frac{v'^2}{c^2}}}\left\{\left[\frac{d^2 r}{dt'^2} - r\left(\frac{d\emptyset}{dt'}\right)^2\right]\hat{r} + \left(2\frac{dr}{dt'}\frac{d\emptyset}{dt'} + r\frac{d^2\emptyset}{dt'^2}\right)\widehat{\emptyset}\right\} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$$

$$\vec{F}'_{\widehat{\emptyset}} = \left(2\frac{dr}{dt'}\frac{d\emptyset}{dt'} + r\frac{d^2\emptyset}{dt'^2}\right)\widehat{\emptyset} = zero \qquad \frac{dL}{dt'} = \frac{d}{dt'}\left(r^2\frac{d\emptyset}{dt'}\right) = 2r\frac{dr}{dt'}\frac{d\emptyset}{dt'} + r^2\frac{d^2\emptyset}{dt'^2} = zero$$

$$\vec{F}'_{\hat{r}} = \frac{m_o}{\sqrt{1+\frac{v'^2}{c^2}}}\left[\frac{d^2 r}{dt'^2} - r\left(\frac{d\emptyset}{dt'}\right)^2\right]\hat{r} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$$

$$\left[\frac{d^2 r}{dt'^2} - r\left(\frac{d\emptyset}{dt'}\right)^2\right] = -\frac{k}{m_o}\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\sqrt{1+\frac{v'^2}{c^2}}$$

$$\left[\frac{d^2 r}{dt'^2} - r\left(\frac{d\emptyset}{dt'}\right)^2\right] = -\frac{k}{m_o}\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\sqrt{1+\frac{v'^2}{c^2}}$$

$$\left[\frac{d^2 r}{dt'^2} - r\left(\frac{d\emptyset}{dt'}\right)^2\right] = -\frac{k}{m_o}\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\left(1 + A\frac{e^{-ar}}{r}\right)$$

$$\frac{d\emptyset}{dt'} = \frac{L'}{r^2} \qquad \frac{dr}{dt'} = -L'\frac{dw}{d\emptyset} \qquad \frac{d^2 r}{dt'^2} = \frac{-L^2}{r^2}\frac{d^2 w}{d\emptyset^2}$$

$$\left[\frac{-L'^2}{r^2}\frac{d^2 w}{d\emptyset^2} - r\left(\frac{L'}{r^2}\right)^2\right] = -\frac{k}{m_o}\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\left(1 + A\frac{e^{-ar}}{r}\right)$$

$$\frac{d^2 w}{d\emptyset^2} + \frac{1}{r} = \frac{k}{m_o L'^2}r^2\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\left(1 + A\frac{e^{-ar}}{r}\right) \qquad B = \frac{k}{m_o L'^2}$$

$$\frac{d^2 w}{d\emptyset^2} + \frac{1}{r} = Br^2\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\left(1 + A\frac{e^{-ar}}{r}\right) \qquad B = \frac{k}{m_o L'^2}$$

$$\frac{d^2 w}{d\emptyset^2} + \frac{1}{r} = Be^{-ar}\left(1 + A\frac{e^{-ar}}{r}\right) + aBre^{-ar}\left(1 + A\frac{e^{-ar}}{r}\right)$$

$$\frac{d^2 w}{d\emptyset^2} + \frac{1}{r} = Be^{-ar} + ABe^{-ar}\frac{e^{-ar}}{r} + aBre^{-ar} + aABre^{-ar}\frac{e^{-ar}}{r}$$

$$\frac{d^2 w}{d\emptyset^2} + \frac{1}{r} = Be^{-ar} + AB\frac{e^{-2ar}}{r} + aBre^{-ar} + aABe^{-2ar} \qquad w = \frac{1}{r} \qquad r = w^{-1}$$

$$\frac{d^2 w}{d\emptyset^2} + w = Be^{-aw^{-1}} + ABwe^{-2aw^{-1}} + aBw^{-1}e^{-aw^{-1}} + aABe^{-2aw^{-1}}$$

$$\frac{d^2 w}{d\emptyset^2} + w = Be^{-aw^{-1}} + aBw^{-1}e^{-aw^{-1}} + ABwe^{-2aw^{-1}} + aABe^{-2aw^{-1}}$$

$$\frac{d^2 w}{d\emptyset^2} + w = (1 + aw^{-1})Be^{-aw^{-1}} + (w + a)ABe^{-2aw^{-1}}$$

$\frac{d^2w}{d\emptyset^2} + \mathrm{w} = \left[(1 + \mathrm{aw}^{-1}) + (\mathrm{w} + \mathrm{a})\mathrm{Ae}^{-\mathrm{aw}^{-1}}\right]\mathrm{Be}^{-\mathrm{aw}^{-1}}$

$\mathrm{r} = \frac{1}{Q\cos(\emptyset)}$ $\quad \mathrm{w} = \frac{1}{r} = Q\cos(\emptyset)$ $\quad \frac{\mathrm{dw}}{d\emptyset} = -Q\mathrm{sen}(\emptyset)$ $\quad \frac{\mathrm{d}^2\mathrm{w}}{d\emptyset^2} = -Q\cos(\emptyset)$

$-Q\cos(\emptyset) + Q\cos(\emptyset) = \left[(1 + \mathrm{aw}^{-1}) + (\mathrm{w} + \mathrm{a})\mathrm{Ae}^{-\mathrm{aw}^{-1}}\right]\mathrm{Be}^{-\mathrm{aw}^{-1}}$

$\mathrm{zero} = \left[(1 + \mathrm{aw}^{-1}) + (\mathrm{w} + \mathrm{a})\mathrm{Ae}^{-\mathrm{aw}^{-1}}\right]\mathrm{Be}^{-\mathrm{aw}^{-1}}$

$(1 + \mathrm{aw}^{-1}) + (\mathrm{w} + \mathrm{a})\mathrm{Ae}^{-\mathrm{aw}^{-1}} = zero$ $\quad \mathrm{w} = \frac{1}{r} = Q\cos(\emptyset)$ $\quad r = \mathrm{w}^{-1} = \frac{1}{Q\cos(\emptyset)}$

$\left[1 + \frac{\mathrm{a}}{Q\cos(\emptyset)}\right] + [Q\cos(\emptyset) + \mathrm{a}]\mathrm{Ae}^{-\mathrm{aw}^{-1}} = zero$

$Q\cos(\emptyset)\left(1 + \frac{\mathrm{a}}{Q\cos(\emptyset)}\right) + Q\cos(\emptyset)[Q\cos(\emptyset) + \mathrm{a}]\mathrm{Ae}^{-\mathrm{aw}^{-1}} = zero$

$Q\cos(\emptyset) + \mathrm{a} + Q^2\cos^2(\emptyset)\mathrm{Ae}^{-\mathrm{aw}^{-1}} + Q\cos(\emptyset)\mathrm{aAe}^{-\mathrm{aw}^{-1}} = zero$

$Q^2\cos^2(\emptyset)\mathrm{Ae}^{-\mathrm{aw}^{-1}} + Q\cos(\emptyset) + Q\cos(\emptyset)\mathrm{aAe}^{-\mathrm{aw}^{-1}} + a = zero$

$Q^2\cos^2(\emptyset)\mathrm{Ae}^{-\mathrm{aw}^{-1}} + Q\cos(\emptyset)\left(1 + \mathrm{aAe}^{-\mathrm{aw}^{-1}}\right) + a = zero$

$$Q\cos(\emptyset) = \frac{-\left(1 + \mathrm{aAe}^{-\mathrm{aw}^{-1}}\right) \pm \sqrt{\left(1 + \mathrm{aAe}^{-\mathrm{aw}^{-1}}\right)^2 - 4\mathrm{Ae}^{-\mathrm{aw}^{-1}}(a)}}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

$$Q\cos(\emptyset) = \frac{-\left(1 + \mathrm{aAe}^{-\mathrm{aw}^{-1}}\right) \pm \sqrt{1 + 2\mathrm{aAe}^{-\mathrm{aw}^{-1}} + \mathrm{a}^2\mathrm{A}^2\mathrm{e}^{-2\mathrm{aw}^{-1}} - 4a\mathrm{Ae}^{-\mathrm{aw}^{-1}}}}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

$$Q\cos(\emptyset) = \frac{-\left(1 - \mathrm{aAe}^{-\mathrm{aw}^{-1}}\right) \pm \sqrt{1 - 2\mathrm{aAe}^{-\mathrm{aw}^{-1}} + \mathrm{a}^2\mathrm{A}^2\mathrm{e}^{-2\mathrm{aw}^{-1}}}}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

$$Q\cos(\emptyset) = \frac{-\left(1 - \mathrm{aAe}^{-\mathrm{aw}^{-1}}\right) \pm \sqrt{\left(1 - \mathrm{aAe}^{-\mathrm{aw}^{-1}}\right)^2}}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

$$Q\cos(\emptyset) = \frac{-\left(1 - \mathrm{aAe}^{-\mathrm{aw}^{-1}}\right) \pm \left(1 - \mathrm{aAe}^{-\mathrm{aw}^{-1}}\right)}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

$$w = \frac{1}{r} = Q\cos(\emptyset) = \frac{-1 + \mathrm{aAe}^{-\mathrm{aw}^{-1}} + 1 - \mathrm{aAe}^{-\mathrm{aw}^{-1}}}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}} = zero$$

$$Q\cos(\emptyset) = \frac{-\left(1 - \mathrm{aAe}^{-\mathrm{aw}^{-1}}\right) \pm \left(1 - \mathrm{aAe}^{-\mathrm{aw}^{-1}}\right)}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

$Q\cos(\emptyset) = \frac{-1+\mathrm{aAe}^{-\mathrm{aw}^{-1}}-1+\mathrm{aAe}^{-\mathrm{aw}^{-1}}}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$ $\quad \mathrm{w} = \frac{1}{r} = Q\cos(\emptyset)$ $\quad r = \mathrm{w}^{-1} = \frac{1}{Q\cos(\emptyset)}$

$$\mathrm{w} = \frac{1}{r} = Q\cos(\emptyset) = \frac{-2 + 2\mathrm{aAe}^{-\mathrm{aw}^{-1}}}{2\mathrm{Ae}^{-\mathrm{aw}^{-1}}} = \frac{-1 + \mathrm{aAe}^{-\mathrm{aw}^{-1}}}{\mathrm{Ae}^{-\mathrm{aw}^{-1}}} = a - \frac{1}{\mathrm{Ae}^{-\mathrm{aw}^{-1}}}$$

§ 29 Yukawa Potential Energy “Continuation”

Newton's gravitational potential energy E_{pN}

$\vec{F} = -\frac{k}{r^2}\hat{r}$ $\quad$ $F = |\vec{F}| = \sqrt{\vec{F}.\vec{F}} = \sqrt{\left(-\frac{k}{r^2}\hat{r}\right)\left(-\frac{k}{r^2}\hat{r}\right)} = \sqrt{\left(\frac{k}{r^2}\right)^2\hat{r}\hat{r}} = \sqrt{\left(\frac{k}{r^2}\right)^2} = \frac{k}{r^2}$

$\vec{F} = -F\hat{r}$ $\quad$ $E_p = -\frac{k}{r}$ $\quad$ $F = \frac{dE_p}{dr} = \frac{k}{r^2}$ $\quad$ k>zero

$\vec{F} = -\frac{dE_p}{dr}\hat{r} = -\frac{k}{r^2}\hat{r}$

Yukawa potential energy E_{pY}

$E_{pY} = -k\frac{e^{-ar}}{r} = -kr^{-1}e^{-ar}$ $\quad$ k>zero $\quad$ a>zero

$$\frac{dE_p}{dr} = \frac{d}{dr}(-kr^{-1}e^{-ar}) = -k\left\{\left[(-1)r^{-1-1}\frac{dr}{dr}\right]e^{-ar} + (r^{-1})e^{-ar}\left(-a\frac{dr}{dr}\right)\right\}$$

$$\frac{dE_p}{dr} = \frac{d}{dr}(-kr^{-1}e^{-ar}) = -k(-r^{-2}e^{-ar} - ar^{-1}e^{-ar}) = k(r^{-2}e^{-ar} + ar^{-1}e^{-ar}) = k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)$$

$$\frac{dE_p}{dr} = \frac{d}{dr}\left(-k\frac{e^{-ar}}{r}\right) = k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)$$

$$E_p = \int dE_p = k\int d\left(-\frac{e^{-ar}}{r}\right) = k\int\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)dr = -k\frac{e^{-ar}}{r}$$

$\vec{F} = -\frac{dE_p}{dr}\hat{r} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$ $\quad$ Attractive force

$$\vec{F} = \frac{m_o\vec{a}}{\sqrt{1-\frac{v^2}{c^2}}} = \vec{F}' = \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'} - v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right] \quad 28.16$$

$$\vec{F} = \frac{m_o\vec{a}}{\sqrt{1-\frac{v^2}{c^2}}} = \vec{F}' = \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'} - v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right] = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$$

$$E_k = \int\vec{F}.d\vec{r} = \int\vec{F}'.d\vec{r} = \int -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}.d\vec{r}$$

$$E_k = \int\frac{m_o\vec{a}}{\sqrt{1-\frac{v^2}{c^2}}}.d\vec{r} = \int\frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'} - v'\frac{dv'}{dt'}\frac{\vec{v}'}{c^2}\right].d\vec{r} = \int -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}.d\vec{r}$$

$$E_k = \int\frac{m_o v dv}{\sqrt{1-\frac{v^2}{c^2}}} = \int\frac{m_o v' dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = \int -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)dr$$

$$dE_k = \vec{F}'.d\vec{r} = \frac{m_o v dv}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_o v' dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)dr \quad 28.17$$

$$E_k = -m_o c^2\sqrt{1-\frac{v^2}{c^2}} = -\frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} = -\left(-k\frac{e^{-ar}}{r}\right) + constante \quad 28.18$$

$$E_R = -m_o c^2\sqrt{1-\frac{v^2}{c^2}} - k\frac{e^{-ar}}{r} = -m_o c^2 \quad 28.18$$

$$E_R = -\frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} - k\frac{e^{-ar}}{r} = -m_o c^2 \qquad \frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = \left(1 - \frac{k}{m_o c^2}\frac{e^{-ar}}{r}\right)^3 = \left(1 - A\frac{e^{-ar}}{r}\right)^3 \quad 28.19$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = \left(1 - A\frac{e^{-ar}}{r}\right)^3 \qquad A = \frac{k}{m_o c^2} \qquad 28.19$$

$$\frac{dE_k}{dt\prime} = \vec{F}'.\frac{d\vec{r}}{dt'} = \frac{m_o v\frac{dv}{dt}}{\sqrt{1-\frac{v^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_o v'\frac{dv'}{dt'}}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\frac{dr}{dt'} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}.\frac{d\vec{r}}{dt\prime}$$

$$\frac{dE_k}{dt\prime} = \vec{F}'.\vec{v}' = \frac{m_o\vec{v}\frac{d\vec{v}}{dt}}{\left(1-\frac{v^2}{c^2}\right)} = \frac{m_o\vec{v}\prime\frac{d\vec{v}\prime}{dt}}{\left(1+\frac{v\prime^2}{c^2}\right)^{\frac{3}{2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}.\vec{v}'$$

$$\frac{dE_k}{dt} = \vec{F}'.\vec{v}' = \frac{m_o\vec{v}.\vec{a}}{\left(1-\frac{v^2}{c^2}\right)} = \frac{m_o\vec{v}\prime.\vec{a}\prime}{\left(1+\frac{v\prime^2}{c^2}\right)^{\frac{3}{2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}.\vec{v}' \qquad 28.20$$

$$\vec{F}' = \frac{m_o\vec{a}\prime}{\left(1+\frac{v\prime^2}{c^2}\right)^{\frac{3}{2}}} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r} \qquad 28.20$$

$$\vec{F}' = \frac{m_o}{\left(1+\frac{v\prime^2}{c^2}\right)^{\frac{3}{2}}}\left\{\left[\frac{d^2r}{dt\prime^2} - r\left(\frac{d\emptyset}{dt\prime}\right)^2\right]\hat{r} + \left(2\frac{dr}{dt\prime}\frac{d\emptyset}{dt\prime} + r\frac{d^2\emptyset}{dt\prime^2}\right)\hat{\emptyset}\right\} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$$

$$\vec{F}'_{\hat{\emptyset}} = \left(2\frac{dr}{dt\prime}\frac{d\emptyset}{dt\prime} + r\frac{d^2\emptyset}{dt\prime^2}\right)\hat{\emptyset} = zero \qquad \frac{dL\prime}{dt\prime} = \frac{d}{dt\prime}\left(r^2\frac{d\emptyset}{dt\prime}\right) = 2r\frac{dr}{dt\prime}\frac{d\emptyset}{dt\prime} + r^2\frac{d^2\emptyset}{dt\prime^2} = zero$$

$$\vec{F}'_{\hat{r}} = \frac{m_o}{\left(1+\frac{v\prime^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{d^2r}{dt\prime^2} - r\left(\frac{d\emptyset}{dt\prime}\right)^2\right]\hat{r} = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)\hat{r}$$

$$\frac{m_o}{\left(1+\frac{v\prime^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{d^2r}{dt\prime^2} - r\left(\frac{d\emptyset}{dt\prime}\right)^2\right] = -k\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)$$

$$\frac{d\emptyset}{dt\prime} = \frac{L\prime}{r^2} \qquad \frac{dr}{dt\prime} = -L'\frac{dw}{d\emptyset} \qquad \frac{d^2r}{dt\prime^2} = \frac{-L'^2}{r^2}\frac{d^2w}{d\emptyset^2}$$

$$\frac{1}{\left(1+\frac{v\prime^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{-L'^2}{r^2}\frac{d^2w}{d\emptyset^2} - r\left(\frac{L\prime}{r^2}\right)^2\right] = -\frac{k}{m_o}\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)$$

$$\frac{1}{\left(1+\frac{v\prime^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{L'^2}{r^2}\frac{d^2w}{d\emptyset^2} + r\left(\frac{L\prime}{r^2}\right)^2\right] = \frac{k}{m_o}\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)$$

$$\frac{1}{\left(1+\frac{v\prime^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{L'^2}{r^2}\frac{d^2w}{d\emptyset^2} + \frac{L'^2}{r^3}\right] = \frac{k}{m_o}\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right)$$

$$\frac{1}{\left(1+\frac{v\prime^2}{c^2}\right)^{\frac{3}{2}}}\left[\frac{d^2w}{d\emptyset^2} + \frac{1}{r}\right] = \frac{k}{m_oL'^2}r^2\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right) \qquad B = \frac{k}{m_oL'^2}$$

$$\frac{1}{\left(1+\frac{v\prime^2}{c^2}\right)^{\frac{3}{2}}}\left(\frac{d^2w}{d\emptyset^2} + \frac{1}{r}\right) = Br^2\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right) \qquad B = \frac{k}{m_oL'^2}$$

$$\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = \left(1 - A\frac{e^{-ar}}{r}\right)^3 = 1 - 3A\frac{e^{-ar}}{r} + 3A^2\frac{e^{-2ar}}{r^2} - A^3\frac{e^{-3ar}}{r^3} \cong 1 - 3A\frac{e^{-ar}}{r}$$

$$3A^2\frac{e^{-2ar}}{r^2} - A^3\frac{e^{-3ar}}{r^3} \cong zero \qquad A = \frac{k}{m_oc^2}$$

$$\left(1 - 3A\frac{e^{-ar}}{r}\right)\left(\frac{d^2w}{d\emptyset^2} + \frac{1}{r}\right) = Br^2\left(\frac{e^{-ar}}{r^2} + a\frac{e^{-ar}}{r}\right) \qquad 28.21$$

$$\left(1-3A\frac{e^{-ar}}{r}\right)\frac{d^2w}{d\emptyset^2}+\left(1-3A\frac{e^{-ar}}{r}\right)\frac{1}{r}=Br^2\left(\frac{e^{-ar}}{r^2}+a\frac{e^{-ar}}{r}\right)$$

$$\frac{d^2w}{d\emptyset^2}-3A\frac{d^2w}{d\emptyset^2}\frac{e^{-ar}}{r}+\frac{1}{r}-3A\frac{e^{-ar}}{r^2}=Br^2\left(\frac{e^{-ar}}{r^2}+a\frac{e^{-ar}}{r}\right)$$

$$\frac{d^2w}{d\emptyset^2}-3A\frac{d^2w}{d\emptyset^2}\frac{e^{-ar}}{r}+\frac{1}{r}-3A\frac{e^{-ar}}{r^2}=Be^{-ar}+raBe^{-ar}$$

$$\frac{d^2w}{d\emptyset^2}\frac{1}{r}-3A\frac{d^2w}{d\emptyset^2}\frac{e^{-ar}}{r^2}+\frac{1}{r^2}-3A\frac{e^{-ar}}{r^3}=B\frac{e^{-ar}}{r}+aBe^{-ar} \qquad w=\frac{1}{r}$$

$$\frac{d^2w}{d\emptyset^2}w-3A\frac{d^2w}{d\emptyset^2}e^{-ar}w^2+w^2-3Ae^{-ar}w^3=Be^{-ar}w+aBe^{-ar}$$

$$\frac{d^2w}{d\emptyset^2}w+w^2=3A\frac{d^2w}{d\emptyset^2}e^{-ar}w^2+3Ae^{-ar}w^3+Be^{-ar}w+aBe^{-ar}$$

$$\frac{d^2w}{d\emptyset^2}w+w^2=e^{-ar}\left(3A\frac{d^2w}{d\emptyset^2}w^2+3Aw^3+Bw+aB\right) \qquad 28.22$$

$$w=\frac{1}{r}=xe^{i\emptyset}+ye^{-i\emptyset} \qquad \frac{dw}{d\emptyset}=ixe^{i\emptyset}-iye^{-i\emptyset} \qquad \frac{d^2w}{d\emptyset^2}=-xe^{i\emptyset}-ye^{-i\emptyset} \qquad i=\sqrt{-1}$$

$$\left(-xe^{i\emptyset}-ye^{-i\emptyset}\right)\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)+\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)^2=e^{-ar}\left(3A\frac{d^2w}{d\emptyset^2}w^2+3Aw^3+Bw+aB\right)$$

$$\left(-xe^{i\emptyset}-ye^{-i\emptyset}\right)\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)=\left[\left(-xe^{i\emptyset}\right)\left(xe^{i\emptyset}\right)+\left(-xe^{i\emptyset}\right)\left(ye^{-i\emptyset}\right)+\left(-ye^{-i\emptyset}\right)\left(xe^{i\emptyset}\right)+\left(-ye^{-i\emptyset}\right)\left(ye^{-i\emptyset}\right)\right]$$
$$=\left(-x^2e^{2i\emptyset}-xy-yx-y^2e^{-2i\emptyset}\right)=-\left(x^2e^{2i\emptyset}+2xy+y^2e^{-2i\emptyset}\right)=-\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)^2$$

$$-\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)^2+\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)^2=e^{-ar}\left(3A\frac{d^2w}{d\emptyset^2}w^2+3Aw^3+Bw+aB\right)$$

$$zero=e^{-ar}\left(3A\frac{d^2w}{d\emptyset^2}w^2+3Aw^3+Bw+aB\right)$$

$$3A\frac{d^2w}{d\emptyset^2}w^2+3Aw^3+Bw+aB=\text{zero}$$

$$3A\left(-xe^{i\emptyset}-ye^{-i\emptyset}\right)\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)^2+3A\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)^3+B\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)+aB=\text{zero}$$

$$3A\left(-xe^{i\emptyset}-ye^{-i\emptyset}\right)\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)^2=-3A\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)^2=-3A\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)^3$$

$$-3A\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)^3+3A\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)^3+B\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)+aB=\text{zero}$$

$$B\left(xe^{i\emptyset}+ye^{-i\emptyset}\right)+aB=\text{zero} \qquad xe^{i\emptyset}+ye^{-i\emptyset}+a=\text{zero}$$

$$w=\frac{1}{r}=xe^{i\emptyset}+ye^{-i\emptyset}=-a$$

In §28 simplified calculation of the perihelion retraction we obtain:

$$E_k = \int \frac{m_o v' dv'}{\sqrt{1+\frac{v'^2}{c^2}}} = \int \frac{m_o v dv}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \int -\frac{k}{r^2} dr \quad dE_k = \vec{F}.d\vec{r} = \frac{m_o v' dv'}{\sqrt{1+\frac{v'^2}{c^2}}} = \frac{m_o v dv}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2} dr \qquad 28.08$$

$$E_k = m_o c^2 \sqrt{1+\frac{v'^2}{c^2}} = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{k}{r} + \text{constant} \qquad E_R = m_o c^2 \sqrt{1+\frac{v'^2}{c^2}} - \frac{k}{r} = m_o c^2 \qquad 28.09$$

$$E_R = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{k}{r} = m_o c^2 \qquad \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \left(1 + \frac{k}{m_o c^2}\frac{1}{r}\right)^3 = \left(1 + A\frac{1}{r}\right)^3 \qquad 28.10$$

In this first variant relativistic kinetic energy is greater than inertial energy $\frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} > m_o c^2$. This causes Mercury's perihelion to recede. The planet seems heavier due to the movement.

$$E_k = \int \frac{m_o v' dv'}{\sqrt{1+\frac{v'^2}{c^2}}} = \int \frac{m_o v dv}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \int -\frac{k}{r^2} dr \qquad 28.08$$

$$E_k = \int_{v'=\text{zero}}^{v'} \frac{m_o v' dv'}{\sqrt{1+\frac{v'^2}{c^2}}} = \int_{v=\text{zero}}^{v} \frac{m_o v dv}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \int_{r=\infty}^{r} -\frac{k}{r^2} dr$$

$$E_k = m_o c^2 \sqrt{1+\frac{v'^2}{c^2}}\Bigg|_{v'=\text{zero}}^{v'} = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}\Bigg|_{v=\text{zero}}^{v} = \frac{k}{r}\Bigg|_{r=\infty}^{r}$$

$$E_k = m_o c^2 \sqrt{1+\frac{v^2}{c^2}} - m_o c^2 \sqrt{1+\frac{(\text{zero})^2}{c^2}} = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{m_o c^2}{\sqrt{1-\frac{(\text{zero})^2}{c^2}}} = \frac{k}{r} - \frac{k}{\infty}$$

$$E_k = m_o c^2 \sqrt{1+\frac{v'^2}{c^2}} - m_o c^2 = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_o c^2 = \frac{k}{r} \qquad m_o c^2 \sqrt{1+\frac{v'^2}{c^2}} \geq m_o c^2 \qquad \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} \geq m_o c^2 \qquad 30.1$$

Defining potential energy as $E_p = -\frac{k}{r}$: 30.2

And applying in 1 we have $E_k = m_o c^2 \sqrt{1+\frac{v'^2}{c^2}} - m_o c^2 = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_o c^2 = -E_p$ 30.3

In 3 we have the energy conservation principle written as $E_k + E_p = \text{zero}$ 30.4

With 3 the kinetic energy equal to $E_k = m_o c^2 \sqrt{1+\frac{v'^2}{c^2}} - m_o c^2 = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_o c^2$ 30.5

n 3 the lowest energy of the system is the inertial energy of rest $E_0 = m_o c^2$ 30.6

In 3 the highest energy of the system is $E = m_o c^2 \sqrt{1+\frac{v'^2}{c^2}} = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}$ 30.7

Now defining $p = \frac{m_o v}{\sqrt{1-\frac{v^2}{c^2}}} = m_o v'$ $vp = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} = m_o v v'$ $T' = m_o c^2 \sqrt{1+\frac{v'^2}{c^2}}$ $T = -m_o c^2 \sqrt{1-\frac{v^2}{c^2}}$ 30.8

And knowing that $E = c\sqrt{m_0^2 c^2 + p^2} = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2 \sqrt{1-\frac{v^2}{c^2}} = vp - T$ 30.9

If in 9 $m_o = \text{zero}$ So $E = cp.$ 30.10

Applying 8 and 9 in 7 we obtain the greatest energy of the written system as:

$E = c\sqrt{m_0^2c^2 + p^2} = T' = vp - T$ This we have $vp = T' + T$ 30.7b

With 8 and 9 we can write 3 in the form:

$$E_k = m_o c^2\sqrt{1 + \frac{v'^2}{c^2}} - m_o c^2 = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2\sqrt{1 - \frac{v^2}{c^2}} - m_o c^2 = -E_p \quad 30.3b$$

$$E_k = T' - E_0 = vp - T - E_0 = -E_p \quad 30.3c$$

From 3c we can define the resting inertial energy $E_0' = m_o$ e $E_0 = m_o$ in the form:

$E_0' = T' + E_p = m_o c^2$ and $E_0 = vp - T + E_p = m_o c^2.$ 30.11

Defining Lagrangean as $L = T - E_p = -m_o c^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{k}{r}$ 30.12

This Lagrangian meets$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$ according to §24.

So from 11 temos: $E_0' = T' + E_p = m_o c^2$ and $E_0 = vp - L = m_o c^2$ 30.13

In §28 simplified calculation of perihelion advance we obtain:

$E_k = \int \frac{m_o v dv}{\sqrt{1-\frac{v^2}{c^2}}} = \int \frac{m_o v' dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = \int -\frac{k}{r^2} dr$ $dE_k = \vec{F}'.d\vec{r} = \frac{m_o v dv}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_o v' dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = -\frac{k}{r^2} dr$ 28.17

$E_k = -m_o c^2\sqrt{1 - \frac{v^2}{c^2}} = -\frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} = \frac{k}{r} + \text{constant}$ $E_R = -m_o c^2\sqrt{1 - \frac{v^2}{c^2}} - \frac{k}{r} = -m_o c^2$ 28.18

$E_R = -\frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} - \frac{k}{r} = -m_o c^2$ $\frac{1}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = \left(1 - \frac{k}{m_o c^2}\frac{1}{r}\right)^3 = \left(1 - A\frac{1}{r}\right)^3$ 28.19

In this second variant relativistic kinetic energy is smaller than inertial energy $\frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} < m_o c^2$. This causes the advance of Mercury's perihelion. The planet really is lighter due to movement.

$$E_k = \int \frac{m_o v dv}{\sqrt{1-\frac{v^2}{c^2}}} = \int \frac{m_o v' dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = \int -\frac{k}{r^2} dr \quad 28.17$$

$$E_k = \int_{v=zero}^{v} \frac{m_o v dv}{\sqrt{1-\frac{v^2}{c^2}}} = \int_{v'=zero}^{v'} \frac{m_o v' dv'}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}} = \int_{r=\infty}^{r} -\frac{k}{r^2} dr$$

$$E_k = -m_o c^2\sqrt{1 - \frac{v^2}{c^2}}\Bigg|_{v=zero}^{v} = -\frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}}\Bigg|_{v'=zero}^{v'} = \frac{k}{r}\Bigg|_{r=\infty}^{r}$$

$$E_k = -m_o c^2\sqrt{1 - \frac{v^2}{c^2}} - \left(-m_o c^2\sqrt{1 - \frac{(zero)^2}{c^2}}\right) = -\frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} - \left(-\frac{m_o c^2}{\sqrt{1+\frac{(zero)^2}{c^2}}}\right) = \frac{k}{r} - \frac{k}{\infty}$$

$$E_k = -m_o c^2\sqrt{1 - \frac{v^2}{c^2}} - (-m_o c^2) = -\frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} - (-m_o c^2) = \frac{k}{r}$$

$$E_k = -m_o c^2 \sqrt{1-\frac{v^2}{c^2}} + m_o c^2 = -\frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} + m_o c^2 = \frac{k}{r}$$

$$E_k = m_o c^2 - m_o c^2\sqrt{1-\frac{v^2}{c^2}} = m_o c^2 - \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} = \frac{k}{r} \qquad m_o c^2 \geq m_o c^2\sqrt{1-\frac{v^2}{c^2}} \qquad m_o c^2 \geq \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} \qquad 30.14$$

Where applying 2 the definition of potential energy $E_p = -\frac{k}{r}$:

We get $E_k = m_o c^2 - m_o c^2\sqrt{1-\frac{v^2}{c^2}} = m_o c^2 - \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} = -E_p$ 30.15

In 15 we have the principle of conservation of energy written as $E_k + E_p = \text{zero}.$

Being 15 the kinetic energy equal to:

$$E_k = m_o c^2 - m_o c^2\sqrt{1-\frac{v^2}{c^2}} = m_o c^2 - \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} \qquad 30.16$$

In 15 the biggest energy of the system is the inertial energy of rest $E_0 = m_o c^2$ 30.17

At 15 the lowest energy in the system is $E' = m_o c^2\sqrt{1-\frac{v^2}{c^2}} = \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}}$ 30.18

Now defining $p' = \frac{m_o v'}{\sqrt{1+\frac{v'^2}{c^2}}} = m_o v$ $v'p' = \frac{m_o v'^2}{\sqrt{1+\frac{v'^2}{c^2}}} = m_o v' v$ 30.19

And knowing that $E' = c\sqrt{m_0^2 c^2 - p'^2} = \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} = -\frac{m_o v'^2}{\sqrt{1+\frac{v'^2}{c^2}}} + m_o c^2\sqrt{1+\frac{v'^2}{c^2}} = -v'p' + T'$ 30.20

If in 20 $m_o = \text{zero}$ so $E' = icp'$ 30.21

Proving 20:

$$E' = \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} = c\sqrt{m_0^2 c^2 - p'^2} = c\sqrt{m_0^2 c^2 - \left(\frac{m_o v'}{\sqrt{1+\frac{v'^2}{c^2}}}\right)^2} = c\sqrt{m_0^2 c^2 - \frac{m_0^2 v'^2}{1+\frac{v'^2}{c^2}}} = m_o c\sqrt{\frac{c^2\left(1+\frac{v'^2}{c^2}\right)}{1+\frac{v'^2}{c^2}} - \frac{v'^2}{1+\frac{v'^2}{c^2}}}$$

$$E' = \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} = c\sqrt{m_0^2 c^2 - p'^2} = \frac{m_o c}{\sqrt{1+\frac{v'^2}{c^2}}}\sqrt{c^2\left(1+\frac{v'^2}{c^2}\right) - v'^2} = \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}}$$

$$E' = m_o c^2\sqrt{1-\frac{v^2}{c^2}} = c\sqrt{m_0^2 c^2 - p'^2} = c\sqrt{m_0^2 c^2 - (m_o v)^2} = c\sqrt{m_0^2 c^2 - m_0^2 v^2} = m_o c\sqrt{c^2 - v^2} = m_o c^2\sqrt{1-\frac{v^2}{c^2}}$$

Applying 8, 19 and 20 to 18 results in the lowest energy of the written system as:

$E' = c\sqrt{m_0^2 c^2 - p'^2} = -T = -v'p' + T'$ This we have $v'p' = T' + T$ 30.18b

With 19 and 20 we can write 15 in the form:

$$E_k = m_o c^2 - m_o c^2\sqrt{1-\frac{v^2}{c^2}} = m_o c^2 + \frac{m_o v'^2}{\sqrt{1+\frac{v'^2}{c^2}}} - m_o c^2\sqrt{1+\frac{v'^2}{c^2}} = -E_p \qquad 30.15b$$

$$E_k = E_0 + T = E_0 + v'p' - T' = -E_p \qquad 30.15c$$

From 15c we can define the resting inertial energy $E'_0 = m_o$ e $E_0 = m_o$ in the form:

$$E_0 = -T - E_p = m_o c^2 \quad \text{and} \quad E'_0 = -v'p' + T' - E_p = m_o c^2. \qquad 30.22$$

Defining Lagrangean as $L' = T' - E_p = m_o c^2\sqrt{1+\frac{v'^2}{c^2}} + \frac{k}{r}$ 30.23

This Lagrangian meets $\frac{d}{dt'}\left(\frac{\partial L'}{\partial \dot{x}'}\right) = \frac{\partial L'}{\partial x}$ prova no final.

$$E_0 = -T - E_p = m_o c^2 \qquad E'_0 = -v'p' + L' = m_o c^2 \qquad 30.24$$

Rewriting 11 e 22:

$$E'_0 = T' + E_p = m_o c^2 \quad \text{and} \quad E_0 = vp - T + E_p = m_o c^2. \qquad 30.11$$

$$E_0 = -T - E_p = m_o c^2 \quad \text{and} \quad E'_0 = -v'p' + T' - E_p = m_o c^2. \qquad 30.22$$

Equating E_0 of 11 with E_0 of 22 we have:

$$E_0 = vp - T + E_p = -T - E_p = m_o c^2$$

This we get $vp = -2E_p$ $\qquad -E_p = \frac{vp}{2} = \frac{1}{2}\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}}$ 30.25

Matching $-E_p = \frac{vp}{2}$ the kinetic energy of 3b we have:

$$E_k = m_o c^2\sqrt{1+\frac{v'^2}{c^2}} - m_o c^2 = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2\sqrt{1-\frac{v^2}{c^2}} - m_o c^2 = -E_p = \frac{vp}{2} = \frac{1}{2}\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad 30.3d$$

In 3d we should have:

$$\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2\sqrt{1-\frac{v^2}{c^2}} - m_o c^2 = \frac{1}{2}\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$m_o v^2 + m_o c^2\left(1-\frac{v^2}{c^2}\right) - m_o c^2\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2} m_o v^2$$

$$\frac{1}{2} m_o v^2 + m_o c^2 - m_o c^2\frac{v^2}{c^2} - m_o c^2\sqrt{1-\frac{v^2}{c^2}} = \text{zero}$$

$$\frac{1}{2} m_o v^2 + m_o c^2 - m_o v^2 - m_o c^2\sqrt{1-\frac{v^2}{c^2}} = \text{zero}$$

$$m_o c^2 - \frac{1}{2}\frac{c^2}{c^2} m_o v^2 - m_o c^2\sqrt{1-\frac{v^2}{c^2}} = \text{zero}$$

$$m_o c^2\left(1-\frac{1}{2}\frac{v^2}{c^2}\right) - m_o c^2\sqrt{1-\frac{v^2}{c^2}} = \text{zero}$$

The approximation $\left(1-\frac{1}{2}\frac{v^2}{c^2}\right) \cong \sqrt{1-\frac{v^2}{c^2}}$ is the cause of Mercury's perihelion setback.

$m_o c^2\sqrt{1-\frac{v^2}{c^2}} - m_o c^2\sqrt{1-\frac{v^2}{c^2}} = \text{zero}$ Result that proves that $E_k = \frac{1}{2}\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}}$.

Equating E'_0 of 11 with E'_0 of 22 we have:

$E'_0 = T' + E_p = -v'p' + T' - E_p = m_o c^2$

This we get $v'p' = -2E_p$ $-E_p = \frac{v'p'}{2} = \frac{1}{2}\frac{m_o v'^2}{\sqrt{1+\frac{v'^2}{c^2}}}$ 30.26

Matching $-E_p = \frac{v'p'}{2}$ the kinetic energy of 15b we have:

$$E_k = m_o c^2 - m_o c^2\sqrt{1-\frac{v^2}{c^2}} = m_o c^2 + \frac{m_o v'^2}{\sqrt{1+\frac{v'^2}{c^2}}} - m_o c^2\sqrt{1+\frac{v'^2}{c^2}} = -E_p = \frac{v'p'}{2} = \frac{1}{2}\frac{m_o v'^2}{\sqrt{1+\frac{v'^2}{c^2}}} \quad 30.15d$$

In 15d we should have:

$$m_o c^2 + \frac{m_o v'^2}{\sqrt{1+\frac{v'^2}{c^2}}} - m_o c^2\sqrt{1+\frac{v'^2}{c^2}} = \frac{1}{2}\frac{m_o v'^2}{\sqrt{1+\frac{v'^2}{c^2}}}$$

$$m_o c^2 + \frac{1}{2}\frac{m_o v'^2}{\sqrt{1+\frac{v'^2}{c^2}}} - m_o c^2\sqrt{1+\frac{v'^2}{c^2}} = \text{zero}$$

$$m_o c^2\sqrt{1+\frac{v'^2}{c^2}} + \frac{1}{2}m_o v'^2 - m_o c^2\left(1+\frac{v'^2}{c^2}\right) = \text{zero}$$

$$m_o c^2\sqrt{1+\frac{v'^2}{c^2}} + \frac{1}{2}m_o v'^2 - m_o c^2 - m_o v'^2 = \text{zero}$$

$$m_o c^2\sqrt{1+\frac{v'^2}{c^2}} - m_o c^2 - \frac{1}{2}m_o \frac{c^2}{c^2} v'^2 = \text{zero}$$

$$m_o c^2\sqrt{1+\frac{v'^2}{c^2}} - m_o c^2\left(1+\frac{1}{2}\frac{v'^2}{c^2}\right) = \text{zero}$$

The approximation $\left(1+\frac{1}{2}\frac{v'^2}{c^2}\right) \cong \sqrt{1+\frac{v'^2}{c^2}}$ is the cause of the advance of Mercury's perihelion.

$m_o c^2\sqrt{1+\frac{v'^2}{c^2}} - m_o c^2\sqrt{1+\frac{v'^2}{c^2}} = \text{zero}$ Result that proves that $E_k = \frac{1}{2}\frac{m_o v'^2}{\sqrt{1+\frac{v'^2}{c^2}}}$.

From 25 and 26 results $vp = v'p'$ 30.27

Applying 25 in E_0 of 22:

$$E_0 = -T - E_p = -T + \frac{vp}{2} = m_o c^2\sqrt{1-\frac{v^2}{c^2}} + \frac{1}{2}\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}\left(1-\frac{2v^2}{2c^2}+\frac{v^2}{2c^2}\right) \quad 30.28$$

$$E_0 = -T - E_p = -T + \frac{vp}{2} = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}\left(1-\frac{v^2}{2c^2}\right) \cong \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}\sqrt{1-\frac{v^2}{c^2}} = m_o c^2 \quad 30.28$$

Applying 25 in E_0 of 11:

$E_0 = vp - T + E_p = \frac{2vp}{2} - T - \frac{vp}{2} = -T + \frac{vp}{2}$ Result already obtained on 28.

Applying 26 in E'_0 of 11

$$E'_0 = T' + E_p = T' - \frac{v'p'}{2} = m_o c^2\sqrt{1+\frac{v'^2}{c^2}} - \frac{1}{2}\frac{m_o v'^2}{\sqrt{1+\frac{v'^2}{c^2}}} = \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}}\left(1+\frac{2v'^2}{2c^2}-\frac{v'^2}{2c^2}\right) \qquad 30.29$$

$$E'_0 = T' + E_p = T' - \frac{v'p'}{2} = \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}}\left(1+\frac{v'^2}{2c^2}\right) \cong \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}}\sqrt{1+\frac{v'^2}{c^2}} = m_o c^2 \qquad 30.29$$

The approximation that exists in 28 and 29 is the cause of the advance and setback of Mercury's perihelion.

Applying 26 in E'_0 of 22

$$E'_0 = -v'p' + T' - E_p = -\frac{2v'p'}{2} + T' + \frac{v'p'}{2} = T' - \frac{v'p'}{2}$$ Result already obtained on 29.

Proof that $\frac{d}{dt'}\left(\frac{\partial L'}{\partial \dot{x}'}\right) = \frac{\partial L'}{\partial x}$ $\qquad L' = T' - E_p = m_o c^2\sqrt{1+\frac{v'^2}{c^2}} + \frac{k}{r}$

$$F'_x = \frac{d}{dt'}\left(\frac{\partial L'}{\partial \dot{x}'}\right) = \frac{\partial L'}{\partial x} \qquad F'_x = \frac{d}{dt'}\left[\frac{\partial}{\partial \dot{x}'}\left(m_o c^2\sqrt{1+\frac{v'^2}{c^2}}\right)\right] = \frac{\partial}{\partial x}\left(\frac{k}{r}\right)$$

$$v' = \frac{ds}{dt'} = \sqrt{\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2} \qquad ds = |ds| = \sqrt{dx^2+dy^2+dz^2} \qquad r^2 = x^2+y^2+z^2$$

$$F'_x = \frac{\partial}{\partial x}\left(\frac{k}{r}\right) = k\frac{\partial}{\partial x}(r^{-1}) = k(-1)r^{-1-1=-2}\frac{\partial r}{\partial x} = -k\frac{1}{r^2}\frac{x}{r} = -k\frac{x}{r^3} \qquad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\vec{F}' = F'_x\hat{\imath} + F'_y\hat{\jmath} + F'_z\hat{k} = -k\frac{x}{r^3}\hat{\imath} - k\frac{y}{r^3}\hat{\jmath} - k\frac{z}{r^3}\hat{k} = -\frac{k}{r^3}\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right) = -\frac{k}{r^3}\vec{r} = -\frac{k}{r^2}\hat{r} \qquad = 19.01$$

$$p'_x = \frac{\partial}{\partial \dot{x}'}\left(m_o c^2\sqrt{1+\frac{v'^2}{c^2}}\right) = m_o c^2\frac{1}{2}\left(1+\frac{v'^2}{c^2}\right)^{\frac{1}{2}-1=-\frac{1}{2}} 2\frac{v'}{c^2}\frac{\partial v'}{\partial \dot{x}'} = \frac{m_o v'}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{\partial v'}{\partial \dot{x}'}$$

$$\frac{\partial v'}{\partial \dot{x}'} = \frac{\partial}{\partial \dot{x}'}\left(\sqrt{\dot{x}'^2+\dot{y}'^2+\dot{z}'^2}\right) = \frac{1}{2}(\dot{x}'^2+\dot{y}'^2+\dot{z}'^2)^{\frac{1}{2}-1=-\frac{1}{2}}2\dot{x}'\frac{\partial \dot{x}'}{\partial \dot{x}'} = \frac{\dot{x}'}{\sqrt{\dot{x}'^2+\dot{y}'^2+\dot{z}'^2}} = \frac{\dot{x}'}{v'}$$

$$p'_x = \frac{\partial}{\partial \dot{x}'}\left(m_o c^2\sqrt{1+\frac{v'^2}{c^2}}\right) = \frac{m_o v'}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{\partial v'}{\partial \dot{x}'} = \frac{m_o v'}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{\dot{x}'}{v'} = \frac{m_o \dot{x}'}{\sqrt{1+\frac{v'^2}{c^2}}}$$

$$F'_x = \frac{dp'_x}{dt'} = \frac{d}{dt'}\left(\frac{m_o\dot{x}'}{\sqrt{1+\frac{v'^2}{c^2}}}\right) = \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)}\left[\frac{d\dot{x}'}{dt'}\sqrt{1+\frac{v'^2}{c^2}} - \dot{x}'\frac{d}{dt'}\left(\sqrt{1+\frac{v'^2}{c^2}}\right)\right] = \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)}\left[\ddot{x}'\sqrt{1+\frac{v'^2}{c^2}} - \dot{x}'\frac{d}{dt'}\left(\sqrt{1+\frac{v'^2}{c^2}}\right)\right]$$

$$\frac{d}{dt'}\left(\sqrt{1+\frac{v'^2}{c^2}}\right) = \frac{1}{2}\left(1+\frac{v'^2}{c^2}\right)^{\frac{1}{2}-1=-\frac{1}{2}} 2\frac{v'}{c^2}\frac{dv'}{dt'} = \frac{\frac{v'dv'}{c^2dt'}}{\sqrt{1+\frac{v'^2}{c^2}}} = \frac{1}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{v'}{c^2}\frac{dv'}{dt'}$$

$$F'_x = \frac{dp'_x}{dt'} = \frac{d}{dt'}\left(\frac{m_o\dot{x}'}{\sqrt{1+\frac{v'^2}{c^2}}}\right) = \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)}\left[\ddot{x}'\sqrt{1+\frac{v'^2}{c^2}} - \dot{x}'\frac{1}{\sqrt{1+\frac{v'^2}{c^2}}}\frac{v'}{c^2}\frac{dv'}{dt'}\right]$$

$$F'_x = \frac{dp'_x}{dt'} = \frac{d}{dt'}\left(\frac{m_o\dot{x}'}{\sqrt{1+\frac{v'^2}{c^2}}}\right) = \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left[\ddot{x}'\left(1+\frac{v'^2}{c^2}\right) - \dot{x}'\frac{v'}{c^2}\frac{dv'}{dt'}\right]$$

$$\vec{F}' = F'_x\hat{\imath} + F'_y\hat{\jmath} + F'_z\hat{k}$$

$$\vec{F}' = \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left\{\left[\ddot{x}'\left(1+\frac{v'^2}{c^2}\right) - \dot{x}'\frac{v'}{c^2}\frac{dv'}{dt'}\right]\hat{\imath} + \left[\ddot{y}'\left(1+\frac{v'^2}{c^2}\right) - \dot{y}'\frac{v'}{c^2}\frac{dv'}{dt'}\right]\hat{\jmath} + \left[\ddot{z}'\left(1+\frac{v'^2}{c^2}\right) - z'\frac{v'}{c^2}\frac{dv'}{dt'}\right]\hat{k}\right\}$$

$$\vec{F}' = \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left\{\ddot{x}'\left(1+\frac{v'^2}{c^2}\right)\hat{\imath} - \dot{x}'\frac{v'}{c^2}\frac{dv'}{dt'}\hat{\imath} + \ddot{y}'\left(1+\frac{v'^2}{c^2}\right)\hat{\jmath} - \dot{y}'\frac{v'}{c^2}\frac{dv'}{dt'}\hat{\jmath} + \ddot{z}'\left(1+\frac{v'^2}{c^2}\right)\hat{k} - z'\frac{v'}{c^2}\frac{dv'}{dt'}\hat{k}\right\}$$

$$\vec{F}' = \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left\{\left(1+\frac{v'^2}{c^2}\right)\left(\ddot{x}'\hat{\imath} + \ddot{y}'\hat{\jmath} + \ddot{z}'\hat{k}\right)\hat{\imath} - \left(\dot{x}' + \dot{y}'\hat{\jmath} + z'\hat{k}\right)\frac{v'}{c^2}\frac{dv'}{dt'}\right\}$$

$$\vec{F}' = \frac{m_o}{\left(1+\frac{v'^2}{c^2}\right)^{\frac{3}{2}}}\left\{\left(1+\frac{v'^2}{c^2}\right)\frac{d\vec{v}'}{dt'} - \frac{v'}{c^2}\frac{dv'}{dt'}\vec{v}'\right\}$$ = 28.16

§30 Energy Continuation Clarifications

With 3d, 6, and 9 we get:

$$\mathbf{E_k = E - E_0 = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_o c^2 = \frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_o c^2\sqrt{1-\frac{v^2}{c^2}} - m_o c^2 = \frac{1}{2}\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{2}vp}$$ 30.30

This we get: $\frac{v^2}{c^2} = \frac{4E_k^2}{c^2p^2}$ 30.31

That applied in 8 results: $p = \frac{m_o v}{\sqrt{1-\frac{4E_k^2}{c^2p^2}}} \rightarrow E_k = \frac{c}{2}\sqrt{p^2 - m_0^2 v^2}$ 30.32

If at 32 $m_o = \text{zero}$ then: $E_k = \frac{cp}{2} = \frac{1}{2}vp \rightarrow v = c$ 30.33

For a particle with velocity $c = \lambda\gamma$ and zero resting mass $m_o = \text{zero}$ we have:

$E = h\gamma$ $\quad p = \frac{h}{\lambda}$ 30.34

Applying $c = \lambda\gamma$ and 34 in 33 results:

$E_k = \frac{cp}{2} = \frac{\lambda\gamma}{2}\frac{h}{\lambda} = \frac{\gamma h}{2} \rightarrow E_k = \frac{h\gamma}{2}$ 30.35

From 34 and 35 we have:

$E = 2E_k$ 30.36

Applying 36 in 30 we have:

$E_k = E - E_0 \rightarrow E_k = 2E_k - E_0 \rightarrow E_0 = E_k = \frac{E}{2}$ 30.37

Applying $E_0 = m_o c^2$ in 37 we obtain:

$E_0 = E_k = \frac{E}{2} = m_o c^2 \rightarrow m_o = \frac{E_k}{c^2} = \frac{E}{2c^2}$ 30.38

With 33 and 38 we get:

$E_k = \frac{cp}{2} = m_o c^2 \rightarrow p = 2m_o c$ 30.39

Clarifications

From 30 and 8 we have $E_k = \frac{1}{2}\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{2}vp$, $E = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}$, $E_0 = m_o c^2$, $p = \frac{m_o v}{\sqrt{1-\frac{v^2}{c^2}}}$ 30.40

Let's apply 40 in energy conservation $E_k = E - E_0$ 30.41

$E_k = E - E_0 \rightarrow \frac{1}{2}\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} - E_0 \rightarrow \frac{1}{2}m_o v^2 = m_o c^2 - E_0\sqrt{1-\frac{v^2}{c^2}}$ 30.42

Doing at 42 $m_o = zero$ we get:

$$\frac{1}{2}(m_o = zero)v^2 = (m_o = zero)c^2 - E_0\sqrt{1-\frac{v^2}{c^2}} \rightarrow -E_0\sqrt{1-\frac{v^2}{c^2}} = zero \qquad 30.43$$

If in 43 $E_0 = zero \rightarrow -(E_0 = zero)\sqrt{1-\frac{v^2}{c^2}} = zero$ without any desirable results.

Now if in 43 $E_0 \neq zero \rightarrow -(E_0 \neq zero)\sqrt{1-\frac{v^2}{c^2}} = zero \rightarrow \frac{v^2}{c^2} = 1 \rightarrow v = c$ 30.44

In 44 we get $v = c$ regardless of the value of $E_0 \neq zero$.

Of 40 we get: $\frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2E_k}{v^2} = \frac{E}{c^2} = \frac{p}{v}$ 30.45

Of 45 we get: $\frac{v^2}{c^2} = \frac{2E_k}{E} = \frac{4E_k^2}{c^2p^2} = \frac{c^2p^2}{E^2}$ 30.46

But $\frac{v^2}{c^2} = 1$ of 44 was obtained from the conservation of energy $E_k = E - E_0$ when $m_o = zero$ and $E_0 \neq zero$ so in 46 we should have $\frac{v^2}{c^2} = \frac{2E_k}{E} = \frac{4E_k^2}{c^2p^2} = \frac{c^2p^2}{E^2} = 1$ 30.47

When we have $m_o = zero$, $v = c$ and $E_0 \neq zero$ out of 47 we get

$\frac{v^2}{c^2} = \frac{2E_k}{E} = 1 \rightarrow E = 2E_k$ equal to 36 30.48

$\frac{v^2}{c^2} = \frac{4E_k^2}{c^2p^2} = 1 \rightarrow E_k = \frac{cp}{2}$ equal to 33 30.49

$\frac{v^2}{c^2} = \frac{c^2p^2}{E^2} = 1 \rightarrow E = cp$ equal to 10 30.50

Applying 9 and 32 to the energy conservation equation $E_k = E - E_0$ we obtain:

$$E_k = E - E_0 \rightarrow \frac{c}{2}\sqrt{p^2 - m_0^2v^2} = c\sqrt{m_0^2c^2 + p^2} - E_0 \qquad 30.51$$

In 51 doing $m_o = zero$ we get:

$\frac{c}{2}\sqrt{p^2 - (m_0^2 = zero)v^2} = c\sqrt{(m_0^2 = zero)c^2 + p^2} - E_0 \rightarrow E_0 = \frac{cp}{2} = E_k$ equal to 37 30.52

Let's start with the equation 8.5:

$$\frac{\partial}{\partial x}+\frac{x/t}{c^2}\frac{\partial}{\partial t}=zero \qquad 8.5$$

$$\frac{\partial}{\partial x}+\frac{x/t}{c^2}\frac{\partial}{\partial t}=\frac{\partial}{\partial x}+\frac{c}{c^2}\frac{\partial}{\partial t}=\frac{\partial}{\partial x}+\frac{1}{c}\frac{\partial}{\partial t}=zero \qquad c=\frac{x}{t} \qquad 31.1$$

$$\frac{\partial}{\partial x}+\frac{1}{c}\frac{\partial}{\partial t}=zero \qquad 31.2$$

The variables involved will be:

$$c=\lambda\gamma \qquad p=\frac{h}{\lambda} \qquad E=h\gamma \qquad K=\frac{2\pi}{\lambda} \qquad \omega=2\pi\gamma \qquad 31.3$$

$$p=\frac{h}{\lambda}=\frac{h}{\frac{2\pi}{K}}=\frac{h}{2\pi}K=\hbar K \qquad E=h\gamma=h\frac{\omega}{2\pi}=\hbar\omega \qquad \hbar=\frac{h}{2\pi} \qquad 31.4$$

$$p=\hbar K \qquad E=\hbar\omega \qquad E=cp \qquad K=\frac{\omega}{c} \qquad \omega=cK \qquad 31.5$$

Function construction Ψ:

$$c=\lambda\gamma=\frac{x}{t}\rightarrow\frac{x}{\lambda}=\gamma t\rightarrow\frac{x}{\lambda}-\gamma t=zero\rightarrow i2\pi\left(\frac{x}{\lambda}-\gamma t\right)=\mathrm{i}\left(\frac{2\pi}{\lambda}x-2\pi\gamma t\right)=\mathrm{i}(Kx-\omega t)=zero \quad 31.6$$

$$\mathrm{e}^{\mathrm{i}(Kx-\omega t)}=\mathrm{e}^{\mathrm{zero}}=1 \qquad \mathrm{i}=\sqrt{-1} \qquad \mathrm{i}^2=-1 \qquad 31.7$$

$$\Psi=\Psi(x,t)=\mathrm{e}^{\mathrm{i}(Kx-\omega t)} \qquad 31.8$$

Some derivatives of the function $\Psi=\mathrm{e}^{\mathrm{i}(Kx-\omega t)}$:

$$\frac{\partial\Psi}{\partial t}=\mathrm{e}^{\mathrm{i}(Kx-\omega t)}(-i\omega)=-\mathrm{i}\omega\Psi \qquad \frac{\partial^2\Psi}{\partial t^2}=(-i\omega)(-i\omega)\mathrm{e}^{\mathrm{i}(Kx-\omega t)}=-\omega^2\Psi$$

$$\frac{\partial\Psi}{\partial t}=-\mathrm{i}\omega\Psi \qquad \frac{\partial^2\Psi}{\partial t^2}=-\omega^2\Psi \qquad 31.9$$

$$\frac{\partial\Psi}{\partial x}=\mathrm{e}^{\mathrm{i}(Kx-\omega t)}iK=iK\Psi \qquad \frac{\partial^2\Psi}{\partial x^2}=\mathrm{e}^{\mathrm{i}(Kx-\omega t)}iKiK=-K^2\Psi$$

$$\frac{\partial\Psi}{\partial x}=iK\Psi \qquad \frac{\partial^2\Psi}{\partial x^2}=-K^2\Psi \qquad 31.10$$

$$d\Psi=\frac{\partial\Psi}{\partial x}\mathrm{dx}+\frac{\partial\Psi}{\partial t}\mathrm{dt}=d(1)=zero\rightarrow iK\Psi\mathrm{dx}-\mathrm{i}\omega\Psi\mathrm{dt}=\mathrm{zero}\rightarrow\frac{\mathrm{dx}}{\mathrm{dt}}=\frac{\omega}{K}=c\rightarrow\frac{\mathrm{dx}}{\mathrm{dt}}=\frac{x}{t} \qquad 1.13$$

Applying the function Ψ in 2 we obtain:

$$\frac{\partial\Psi}{\partial x}+\frac{1}{c}\frac{\partial\Psi}{\partial t}=zero \qquad 31.11$$

$$\frac{\partial\Psi}{\partial x}+\frac{1}{c}\frac{\partial\Psi}{\partial t}=iK\Psi-\frac{1}{c}\mathrm{i}\omega\Psi=\mathrm{zero} \qquad K=\frac{\omega}{c}$$

Construction of the wave equation:

$$\left(\frac{\partial}{\partial x}=-\frac{1}{c}\frac{\partial}{\partial t}\right)x\left(\frac{\partial\Psi}{\partial x}=-\frac{1}{c}\frac{\partial\Psi}{\partial t}\right)\rightarrow\frac{\partial^2\Psi}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}\rightarrow-K^2\Psi=-\frac{\omega^2}{c^2}\Psi\rightarrow K=\frac{\omega}{c} \qquad 31.12$$

$\frac{\partial^2\Psi}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}=zero$ Where we have $\Psi=\Psi(x,t)$. This is the wave equation 31.13

Construction of the first Erwin Schrödinger equation using the wave equation:

$$\frac{\partial^2\Psi}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}=zero\rightarrow\frac{\partial^2\Psi}{\partial x^2}-\frac{1}{c^2}(-\omega^2\Psi)=zero\rightarrow\frac{\partial^2\Psi}{\partial x^2}+K^2\Psi=zero \qquad 31.14$$

$$\frac{\partial^2\Psi}{\partial x^2}+K^2\Psi=zero \qquad \Psi=\Psi(x) \qquad \frac{\partial\Psi}{\partial x}=\frac{d\Psi}{dx} \qquad 31.15$$

$\frac{d^2\Psi}{dx^2} + K^2\Psi = zero$ 31.16

$\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + \frac{\hbar^2}{2m}K^2\Psi = zero$ $\quad -\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} - \frac{\hbar^2}{2m}K^2\Psi = zero$ 31.17

$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} - \frac{\hbar^2}{2m}K^2\Psi = -\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} - \frac{p^2}{2m}\Psi = zero$ $\quad p = \hbar K$ 31.18

If at 30.4 we have $E_k + E_p(x) \neq zero$ then we can write $E = E_k + E_p(x) = \hbar\omega$. 31.19

Erwin Schrödinger adopted for energy $E = \frac{p^2}{2m} + E_p(x)$ where we have $E_k = \frac{p^2}{2m}$. 31.20

Of 20 we get $\quad -\frac{p^2}{2m} = E_p(x) - E$ $\quad$ that applied in 18 results in: 31.21

$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} - \frac{p^2}{2m}\Psi = -\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + [E_p(x) - E]\Psi = zero$ 31.22

$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + E_p(x)\Psi = E\Psi$ $\quad$ In this we have: $\Psi = \Psi(x)$ 31.23

This 23 is Erwin Schrödinger's equation independent of time for single dimension.

Construction of the second Erwin Schrödinger equation using equations 14 and 11:

Multiplying 14 by $\frac{\hbar^2}{2m}$ gives $\frac{\hbar^2}{2m}$: $\quad \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{\hbar^2}{2m}K^2\Psi = zero$ 31.24

Of 11 we get: $\quad \frac{\partial\Psi}{\partial x} + \frac{1}{c}\frac{\partial\Psi}{\partial t} = zero \rightarrow c\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial t} = zero \rightarrow ciK\Psi + \frac{\partial\Psi}{\partial t} = i\omega\Psi + \frac{\partial\Psi}{\partial t} = zero$

$i\omega\Psi + \frac{\partial\Psi}{\partial t} = zero \rightarrow ii\hbar\omega\Psi + i\hbar\frac{\partial\Psi}{\partial t} = zero \rightarrow -\hbar\omega\Psi + i\hbar\frac{\partial\Psi}{\partial t} = zero$ 31.25

Adding 24 and 25 = $\quad \left(\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + \frac{\hbar^2}{2m}K^2\Psi = zero\right)$ $\quad + \quad \left(-\hbar\omega\Psi + i\hbar\frac{\partial\Psi}{\partial t} = zero\right)$

$\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{\hbar^2}{2m}K^2\Psi - \hbar\omega\Psi + i\hbar\frac{\partial\Psi}{\partial t} = zero$ 31.26

$\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{\hbar^2}{2m}K^2\Psi - \hbar\omega\Psi = -i\hbar\frac{\partial\Psi}{\partial t}$ 31.27

$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{\hbar^2}{2m}K^2\Psi + \hbar\omega\Psi = i\hbar\frac{\partial\Psi}{\partial t}$ 31.28

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \hbar\omega\Psi - \frac{\hbar^2}{2m}K^2\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \left(\hbar\omega - \frac{\hbar^2}{2m}K^2\right)\Psi = i\hbar\frac{\partial\Psi}{\partial t}$ $\quad E = \hbar\omega$ $\quad p^2 = \hbar^2K^2$ 31.29

$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \left(E - \frac{p^2}{2m}\right)\Psi = i\hbar\frac{\partial\Psi}{\partial t}$ 31.30

From the energy of Erwin Schrödinger we obtain $E = \frac{p^2}{2m} + E_p(x) \rightarrow E_p(x) = E - \frac{p^2}{2m}$ 31.31

Applying 31 out of 30 We get:

$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + E_p(x)\Psi = i\hbar\frac{\partial\Psi}{\partial t}$ $\quad$ In this we have: $\Psi = \Psi(x, t)$ 31.32

This 32 is Erwin Schrödinger's equation dependent on space and time.

From 30.8 and 30.3d we get:

$$\mathrm{p} = \frac{\mathrm{m_o v}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} \quad 31.33$$

$$\mathrm{E_k} = \frac{\mathrm{m_o v^2}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} + \mathrm{m_o c^2}\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}} - \mathrm{m_o c^2} = \frac{1}{2}\frac{\mathrm{m_o v^2}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} \quad 31.34$$

If in both equations in the $\frac{\mathrm{v}}{\mathrm{c}}$ ratio the speed of light is considered to be infinite, then we will have $\frac{\mathrm{v}}{\mathrm{c}=\infty} = \mathrm{zero}$ resulting in:

$$\mathrm{p} = \mathrm{m_o v} \quad 31.35$$

$$\mathrm{m_o v^2} + \mathrm{m_o c^2} - \mathrm{m_o c^2} = \frac{1}{2}\mathrm{m_o v^2} \rightarrow \mathrm{E_k} = \frac{1}{2}\mathrm{m_o v^2} \quad 31.36$$

This is what happens in Quantum Mechanics, the speed of light has the character of being infinite and therefore Erwin Schrödinger's energy equation $E = E_k + E_p(x)$ where we have $E_k = \frac{p^2}{2m}$ presents perfect results. We should note that at 36 the inertial energy $\mathrm{m_o c^2}$ also disappears.

Function construction Ψ:

$$c = \frac{E}{p} = \frac{x}{t} \rightarrow px = Et \rightarrow px - Et = zero \rightarrow \frac{i}{\hbar}(px - Et) = zero \quad 31.37$$

$$\mathrm{e}^{\frac{i}{\hbar}(px-Et)} = \mathrm{e}^{\mathrm{zero}} = 1 \qquad \mathrm{i} = \sqrt{-1} \qquad \mathrm{i}^2 = -1 \quad 31.38$$

$$\Psi = \Psi(x,t) = \mathrm{e}^{\frac{i}{\hbar}(px-Et)} \quad 31.39$$

Some derivatives of the function $\Psi = \mathrm{e}^{\frac{i}{\hbar}(px-Et)}$:

$$\frac{\partial\Psi}{\partial t} = \mathrm{e}^{\frac{i}{\hbar}(px-Et)}\left(-\frac{i}{\hbar}E\right) = -\frac{i}{\hbar}E\Psi \qquad \frac{\partial\Psi}{\partial t} = -\frac{i}{\hbar}E\Psi \qquad \mathrm{E}\Psi = -\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} \quad 31.40$$

$$\frac{\partial^2\Psi}{\partial t^2} = \mathrm{e}^{\frac{i}{\hbar}(px-Et)}\left(-\frac{i}{\hbar}E\right)\left(-\frac{i}{\hbar}E\right) = -\frac{1}{\hbar^2}E^2\Psi \qquad \frac{\partial^2\Psi}{\partial t^2} = -\frac{1}{\hbar^2}E^2\Psi \qquad E^2\Psi = -\hbar^2\frac{\partial^2\Psi}{\partial t^2} \quad 31.41$$

$$\frac{\partial\Psi}{\partial x} = \mathrm{e}^{\frac{i}{\hbar}(px-Et)}\left(\frac{i}{\hbar}p\right) = \frac{i}{\hbar}p\Psi \qquad \frac{\partial\Psi}{\partial x} = \frac{i}{\hbar}p\Psi \qquad \mathrm{p}\Psi = \frac{\hbar}{i}\frac{\partial\Psi}{\partial x} \quad 31.42$$

$$\frac{\partial^2\Psi}{\partial x^2} = \mathrm{e}^{\frac{i}{\hbar}(px-Et)}\left(\frac{i}{\hbar}p\right)\left(\frac{i}{\hbar}p\right) = -\frac{1}{\hbar^2}p^2\Psi \qquad \frac{\partial^2\Psi}{\partial x^2} = -\frac{1}{\hbar^2}p^2\Psi \qquad p^2\Psi = -\hbar^2\frac{\partial^2\Psi}{\partial x^2} \quad 31.43$$

From the total differential of Ψ we obtain:

$$d\Psi = \frac{\partial\Psi}{\partial x}\mathrm{dx} + \frac{\partial\Psi}{\partial t}\mathrm{dt} = d(1) = zero \rightarrow \left(\frac{i}{\hbar}p\Psi\right)\mathrm{dx} + \left(-\frac{i}{\hbar}E\Psi\right)\mathrm{dt} = \mathrm{zero} \rightarrow \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{E}{p} = c \rightarrow \frac{\mathrm{dx}}{\mathrm{dt}} = \frac{x}{t} \quad 31.44$$

Applying the function Ψ and its derivatives in $E = cp$ and $E^2 = c^2p^2$ we obtain 31.11 and 31.13:

$$E\Psi = cp\Psi \rightarrow -\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} = c\frac{\hbar}{i}\frac{\partial\Psi}{\partial x} \rightarrow -\frac{\partial\Psi}{\partial t} = c\frac{\partial\Psi}{\partial x} \rightarrow \frac{\partial\Psi}{\partial x} + \frac{1}{c}\frac{\partial\Psi}{\partial t} = zero \qquad = 31.11 \quad 31.45$$

$$E^2\Psi = c^2p^2\Psi \rightarrow -\hbar^2\frac{\partial^2\Psi}{\partial t^2} = c^2\left(-\hbar^2\frac{\partial^2\Psi}{\partial x^2}\right) \rightarrow \frac{\partial^2\Psi}{\partial t^2} = c^2\frac{\partial^2\Psi}{\partial x^2} \rightarrow \frac{\partial^2\Psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = zeo \qquad = 31.13 \quad 31.45b$$

Let's write Erwin Schrödinger's energy equation $E = E_k + E_p(x)$ where we have $E_k = \frac{p^2}{2m}$:

$$E = E_k + E_p(x) = \frac{p^2}{2m} + E_p(x) \rightarrow \frac{p^2}{2m} + E_p(x) = E \quad 31.46$$

In this we apply the function Ψ and its derivatives:

$$\frac{p^2\Psi}{2m} + E_p(x)\Psi = E\Psi \rightarrow \frac{1}{2m}\left(-\hbar^2\frac{\partial^2\Psi}{\partial x^2}\right) + E_p(x)\Psi = E\Psi \quad 31.47$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + E_p(x)\Psi = E\Psi \qquad \text{In this } \Psi = \Psi(x) \rightarrow \frac{\partial^2\Psi}{\partial x^2} = \frac{d^2\Psi}{dx^2} \quad 31.48$$

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + E_p(x)\Psi = E\Psi \quad 31.49$$

This is 49 is Erwin Schrödinger's equation independent of time for a single dimension.

Let's write Erwin Schrödinger's energy equation $E = E_k + E_p(x)$ again where we have $E_k = \frac{p^2}{2m}$:

$$E = E_k + E_p(x) = \frac{p^2}{2m} + E_p(x) \rightarrow \frac{p^2}{2m} + E_p(x) = E \quad 31.50$$

In this we apply the function Ψ and its derivatives:

$$\frac{p^2\Psi}{2m} + E_p(x)\Psi = E\Psi \rightarrow \frac{1}{2m}\left(-\hbar^2\frac{\partial^2\Psi}{\partial x^2}\right) + E_p(x)\Psi = -\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} \quad 31.51$$

$$\frac{1}{2m}\left(-\hbar^2\frac{\partial^2\Psi}{\partial x^2}\right) + E_p(x)\Psi = -\frac{i.\hbar}{i.i}\frac{\partial\Psi}{\partial t} \rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + E_p(x)\Psi = i\hbar\frac{\partial\Psi}{\partial t} \quad 31.52$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + E_p(x)\Psi = i\hbar\frac{\partial\Psi}{\partial t} \qquad \text{In this we have: } \Psi = \Psi(x,t) \quad 31.53$$

This 53 is Erwin Schrödinger's equation dependent on space and time.

A particle moving with velocity v along the x axis is associated with an infinite wave in the form:

$$\Psi = \Psi(x,t) = Ae^{\frac{i}{\hbar}\emptyset} = Ae^{\frac{i}{\hbar}(px-Et)} \qquad A = \text{Constant} \qquad 32.1$$

For a plane wave of constant phase $\emptyset = \emptyset(x,t) = px - Et = \text{constant}$ we obtain the velocity u of phase equal to:

$$d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial t}dt = \text{zero} \rightarrow d\emptyset = pdx - Edt = \text{zero} \rightarrow u = \frac{dx}{dt} = \frac{E}{p} \qquad u = \frac{E}{p} \qquad 32.2$$

The energy E, and the moment p being properties of a particle in motion with velocity v, and the frequency γ and wavelength λ being properties of the wave motion associated with the particle. Louis De Broglie listed these properties in the following equations

$$E = h\gamma = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad p = hk = \frac{m_o v}{\sqrt{1-\frac{v^2}{c^2}}} \qquad k = \frac{1}{\lambda} \qquad 32.3$$

From 3 we get the phase speed u:

$$u = \frac{\gamma}{k} = \frac{E}{p} = \frac{c^2}{v} \qquad 32.4$$

In 4 we have $m_0 > \text{zero}$ because if $m_0 = \text{zero}$ then $E = cp$ (30.10) and we would have:

$$u = \frac{\gamma}{k} = \frac{E}{p} = c \qquad 32.5$$

And in 4 the phase velocity would be $u = c$ and not $u = \frac{c^2}{v}$. 32.6

As $m_0 > \text{zero}$ then $v < c$ and in 4 we have $u > c$. 32.7

If at 4 $u = \frac{c^2}{v}$ the phase velocity then $c \neq \frac{\gamma}{k}$ because if at 4 $c = \frac{\gamma}{k}$ then we would have:

$$u = \frac{\gamma}{k} = \frac{E}{p} = \frac{c^2}{v} = \frac{\left(\frac{\gamma}{k}\right)^2}{v} \rightarrow \frac{\gamma}{k} = \frac{\left(\frac{\gamma}{k}\right)^2}{v} \rightarrow u = v = c = \frac{\gamma}{k}. \qquad 32.8$$

And in 4 the phase velocity would be $u = v = c$ and not $u = \frac{c^2}{v}$. 32.9

From 4 we get the velocity v written as:

$$u = \frac{\gamma}{k} = \frac{E}{p} = \frac{c^2}{v} \rightarrow v = c^2\frac{p}{E} \qquad 32.10$$

Applying 10 in kinetic energy $E_k = \frac{1}{2}vp = \frac{1}{2}\frac{m_o v^2}{\sqrt{1-\frac{v^2}{c^2}}}$ we get to $m_0 > \text{zero}$ e $v < c$:

$$E_k = \frac{1}{2}vp = \frac{1}{2}\left(c^2\frac{p}{E}\right)p = \frac{1}{2}\frac{c^2p^2}{E} \rightarrow E_k = \frac{1}{2}\frac{c^2p^2}{E} \qquad 32.11$$

When $m_0 = \text{zero}$ then $v = c$ and we have $E_k = \frac{cp}{2}$ (30.33) and $E = cp$ (30.10).

Multiplying 30.33 by 30.10 we obtain:

$$E_kE = \frac{cp}{2}cp \rightarrow E_k = \frac{1}{2}\frac{c^2p^2}{E} \qquad 32.12$$

And we have 11 equal to 12 demonstrating that the equation $E_k = \frac{1}{2}\frac{c^2p^2}{E}$ is ambivalent and has general validity for $m_0 \geq \text{zero}$ and $v \leq c$.

We know from mathematics that the group velocity v_g is given by: $v_g = \frac{d\gamma}{dk}$ 32.13

From 3 we get the speed in the form:

$$E = h\gamma = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} \rightarrow (h\gamma)^2 \left(1 - \frac{v^2}{c^2}\right) = m_0^2 c^4 \rightarrow \frac{v^2}{c^2} = 1 - \frac{m_0^2 c^4}{h^2 \gamma^2} \rightarrow v = c\sqrt{1 - \frac{m_0^2 c^4}{h^2 \gamma^2}} \qquad 32.14$$

In 14 we have the particle velocity only as a function of the frequency $v = v(\gamma)$.

Deriving the velocity of 14 in relation to the frequency we obtain:

$$\frac{v^2}{c^2} = 1 - \frac{m_0^2 c^4}{h^2 \gamma^2} \rightarrow \frac{v^2}{c^2} = 1 - \frac{m_0^2 c^4}{h^2} \gamma^{-2} \rightarrow \frac{2v}{c^2}\frac{dv}{d\gamma} = -\frac{m_0^2 c^4}{h^2}(-2)\gamma^{-3} \rightarrow \frac{dv}{d\gamma} = \frac{c^2}{v}\left(\frac{m_0^2 c^4}{h^2 \gamma^3}\right)$$

$$\frac{dv}{d\gamma} = \frac{c^2}{v}\left(\frac{m_0^2 c^4}{h^2 \gamma^3}\right) = \frac{\gamma}{k}\left(\frac{m_0^2 c^4}{h^2 \gamma^3}\right) = \frac{m_0^2 c^4}{h^2 k \gamma^2} \rightarrow \frac{dv}{d\gamma} = \frac{m_0^2 c^4}{h^2 k \gamma^2} \qquad 32.15$$

Deriving the velocity v from 10 in relation to the frequency and considering that k is a function of the frequency $k = k(\gamma)$ we obtain:

$$v = c^2 \frac{p}{E} = c^2 \frac{k}{\gamma} = c^2 k \gamma^{-1} \rightarrow \frac{dv}{d\gamma} = c^2 \left[\frac{dk}{d\gamma}\gamma^{-1} + k(-1)\gamma^{-1-1}\right] = c^2 \left(\frac{1}{\gamma}\frac{dk}{d\gamma} - \frac{k}{\gamma^2}\right) \rightarrow \frac{dv}{d\gamma} = c^2 \left(\frac{1}{\gamma}\frac{dk}{d\gamma} - \frac{k}{\gamma^2}\right) \qquad 32.16$$

We should have 15 equals 16 so:

$$\frac{dv}{d\gamma} = \frac{m_0^2 c^4}{h^2 k \gamma^2} = c^2 \left(\frac{1}{\gamma}\frac{dk}{d\gamma} - \frac{k}{\gamma^2}\right) \rightarrow \frac{m_0^2 c^2}{h^2 k \gamma^2} = \frac{1}{\gamma}\frac{dk}{d\gamma} - \frac{k}{\gamma^2} \rightarrow \frac{m_0^2 c^2}{h^2 k \gamma} = \frac{dk}{d\gamma} - \frac{k}{\gamma} \rightarrow \frac{dk}{d\gamma} = \frac{m_0^2 c^2}{h^2 k \gamma} + \frac{k}{\gamma}$$

$$\frac{dk}{d\gamma} = \frac{m_0^2 c^2}{h^2 k \gamma} + \frac{k}{\gamma} \rightarrow \frac{dk}{d\gamma} = \frac{m_0^2 c^2}{h^2 k \gamma} + \frac{h^2 k k}{h^2 k \gamma} = \frac{m_0^2 c^2 + h^2 k^2}{h^2 k \gamma} = \frac{m_0^2 c^2 + p^2}{Ep} = \frac{\frac{E^2}{c^2}}{Ep} = \frac{1}{c^2}\frac{E}{p}$$

$$\frac{dk}{d\gamma} = \frac{1}{c^2}\frac{E}{p} = \frac{1}{v} \rightarrow v_g = \frac{d\gamma}{dk} = v \rightarrow v_g = v \qquad 32.17$$

And at 17 we have the group velocity v_g equal to the velocity v of the particle.

From 30.9 we obtain:

$$E = c\sqrt{m_0^2 c^2 + p^2} \rightarrow \frac{E^2}{c^2} = p^2 + m_0^2 c^2 \qquad 32.18$$

Applying 3 out of 18 and deriving the frequency γ with respect to k we obtain:

$$\frac{E^2}{c^2} = p^2 + m_0^2 c^2 \rightarrow \frac{h^2 \gamma^2}{c^2} = h^2 k^2 + m_0^2 c^2 \rightarrow \frac{h^2}{c^2} 2\gamma \frac{d\gamma}{dk} = h^2 2k \rightarrow \frac{d\gamma}{dk} = c^2 \frac{k}{\gamma} \qquad 32.19$$

$$v_g = \frac{d\gamma}{dk} = c^2 \frac{k}{\gamma} = v \rightarrow v_g = v \qquad 32.20$$

And in 20 we have the group velocity v_g equal to the velocity v of the particle.

The equation $E = E_k + E_p = \frac{p^2}{2m} + E_p$ by Erwin Schödinger of Quantum Mechanics equals the total energy E with the sum of the kinetic energy E_k with the potential energy E_p functions, to proceed with this recipe it is necessary to name some functions.

The name of kinetic energy in relativity should only be attributed to differences between energies, the best examples are:

$$E_k = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_o c^2 = E - E_0 \qquad 32.21$$

$$E'_k = m_o c^2 - \frac{m_o c^2}{\sqrt{1+\frac{v'^2}{c^2}}} = E_0 - E' \qquad 32.22$$

Writing 30.3:

$$E_k = m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} - m_o c^2 = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_o c^2 = -E_p \qquad 30.3$$

In this denominating T'_k the kinetic energy:

$$T'_k = m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} - m_o c^2 \qquad 32.23$$

And it remains as kinetic energy the term $E_k = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_o c^2$ 32.21

In 30.3 we have the exact result: $T'_k = E_k$ 32.24

The result is exact because applying $\sqrt{1 - \frac{v^2}{c^2}} \sqrt{1 + \frac{v'^2}{c^2}} = 1$ in either one we get the other.

And we have 30.3 written as: $T'_k = E_k = -E_p$ 32.25

Writing 30.15:

$$E_k = m_o c^2 - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = m_o c^2 - \frac{m_o c^2}{\sqrt{1 + \frac{v'^2}{c^2}}} = -E_p \qquad 30.15$$

In this denominating the kinetic energies:

$$T_k = m_o c^2 - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} \quad \text{and} \quad E'_k = m_o c^2 - \frac{m_o c^2}{\sqrt{1 + \frac{v'^2}{c^2}}} \qquad 3\ 2.26$$

At 30.15 we have the exact result: $T_k = E'_k$ 32.27

The result is accurate because applyin $\sqrt{1 - \frac{v^2}{c^2}} \sqrt{1 + \frac{v'^2}{c^2}} = 1$ in either one we get the other.

And we have 30.15 written as: $T_k = E'_k = -E_p$ 32.28

From the kinetic energy 21 we obtain:

$$E_k = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_o c^2 = \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} - m_o c^2 \qquad 32.29$$

$$vp = \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(\frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_o c^2 \right) + \left(m_o c^2 - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} \right) = E_k + T_k \rightarrow vp = E_k + T_k \quad 32.30$$

$$vp = \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_o c^2 \sqrt{1 - \frac{v^2}{c^2}} = E + T \rightarrow vp = E + T \qquad 32.31$$

In this vp is the difference between the highest and lowest energy. Therefore, the average kinetic energy $E_k = \frac{1}{2} vp = \frac{1}{2} \frac{m_o v^2}{\sqrt{1 - \frac{v^2}{c^2}}}$ is the average energy of the difference between the highest and the lowest energy.

From the kinetic energy 22 we obtain:

$$E'_k = m_o c^2 - \frac{m_o c^2}{\sqrt{1 + \frac{v'^2}{c^2}}} = m_o c^2 + \frac{m_o v'^2}{\sqrt{1 + \frac{v'^2}{c^2}}} - m_o c^2 \sqrt{1 + \frac{v'^2}{c^2}} \qquad 32.32$$

$$\mathrm{v'p'} = \frac{\mathrm{m_o v'^2}}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}} = \left(\mathrm{m_o c^2}\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}} - \mathrm{m_o c^2}\right) + \left(\mathrm{m_o c^2} - \frac{\mathrm{m_o c^2}}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}}\right) = T'_k + E'_k \rightarrow \mathrm{v'p'} = T'_k + E'_k \qquad 32.33$$

$$\mathrm{v'p'} = \frac{\mathrm{m_o}\nu'^2}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}} = \mathrm{m_o c^2}\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}} - \frac{\mathrm{m_o c^2}}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}} = \mathrm{T' - E'} \rightarrow \mathrm{v'p' = T' - E'} \qquad 32.34$$

In this v'p' it is the difference between the highest and lowest energy. Therefore, the average kinetic energy $E'_k = \frac{1}{2}\mathrm{v'p'} = \frac{1}{2}\frac{\mathrm{m_o v\prime^2}}{\sqrt{1+\frac{\mathrm{v\prime^2}}{\mathrm{c^2}}}}$ is the average energy of the difference between the highest and the lowest energy.

Comparing 30 with 33 we see that all terms in the sequence are exactly the same so we have:

$$\mathrm{vp = v'p'} \qquad \mathrm{E_k} = T'_k \qquad \mathrm{T_k} = E'_k \qquad 32.35$$

Comparing 31 with 34 we see that all terms in the sequence are exactly the same so we have:

$$\mathrm{vp = v'p'} \qquad \mathrm{E} = T' \qquad \mathrm{T} = -E' \qquad 32.36$$

The energies $\mathrm{E_k}$ and $\mathrm{T_k}$ are related by:

$$\mathrm{E_k} = \frac{\mathrm{m_o c^2}}{\sqrt{1-\frac{\mathrm{v^2}}{\mathrm{c^2}}}} - \mathrm{m_o c^2} = \frac{\mathrm{m_o c^2 - m_o c^2}\sqrt{1-\frac{\mathrm{v^2}}{\mathrm{c^2}}}}{\sqrt{1-\frac{\mathrm{v^2}}{\mathrm{c^2}}}} \qquad \mathrm{E_k} = \frac{\mathrm{T_k}}{\sqrt{1-\frac{\mathrm{v^2}}{\mathrm{c^2}}}} \qquad 32.37$$

The energies E'_k and T'_k are related by:

$$E'_k = \mathrm{m_o c^2} - \frac{\mathrm{m_o c^2}}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}} = \frac{\mathrm{m_o c^2}\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}} - \mathrm{m_o c^2}}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}} \qquad E'_k = \frac{T'_k}{\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}}} \qquad 32.38$$

From 25 and 28 we get:

$$\mathrm{E_k = -E_p \rightarrow E_k + E_p} = zero \qquad E'_k = -\mathrm{E_p} \rightarrow E'_k + \mathrm{E_p} = zero \qquad 32.39$$

Now naming the Hamiltonians H and H' as:

$$\mathrm{H = E_k + E_p} \qquad H' = E'_k + \mathrm{E_p} \qquad 32.40$$

The Hamiltonians being the total energy of the particle, which by hypothesis is not necessarily equal to zero.

Now we must also define the Lagrangian in terms of kinetic energy.

Now from 30 and 33 we get:

$$\mathrm{vp = E_k + T_k \rightarrow E_k = vp - T_k} \qquad \mathrm{v'p' = T'_k + E'_k \rightarrow E'_k = v'p' - T'_k} \qquad 32.41$$

Applying 41 out of 40 we get:

$$\mathrm{H = E_k + E_p = vp - T_k + E_p} \qquad H' = E'_k + \mathrm{E_p} = \mathrm{v'p'} - T'_k + \mathrm{E_p} \qquad 32.42$$

Defining Lagrangian as:

$$\mathrm{L = T_k - E_p = m_o c^2 - m_o c^2}\sqrt{1-\frac{\mathrm{v^2}}{\mathrm{c^2}}} + \frac{\mathrm{k}}{\mathrm{r}} \qquad 32.43$$

$$L' = T'_k - \mathrm{E_p} = \mathrm{m_o c^2}\sqrt{1+\frac{\mathrm{v'^2}}{\mathrm{c^2}}} - \mathrm{m_o c^2} + \frac{k}{r} \qquad 32.44$$

Applying 43 and 44 to 42 we obtain the relationship between the Hamiltonians and Lagrangians:

$$\mathrm{H} = \mathrm{vp} - \mathrm{T_k} + \mathrm{E_p} = \mathrm{vp} - \left(\mathrm{T_k} - \mathrm{E_p}\right) = \mathrm{vp} - \mathrm{L} \rightarrow \mathrm{H} = \mathrm{vp} - \mathrm{L} \qquad 32.45$$

$$\mathrm{H'} = \mathrm{v'p'} - \mathrm{T'_k} + \mathrm{E_p} = \mathrm{v'p'} - \left(\mathrm{T'_k} - \mathrm{E_p}\right) = \mathrm{v'p'} - \mathrm{L'} \rightarrow \mathrm{H'} = \mathrm{v'p'} - \mathrm{L'} \qquad 32.46$$

Now to redefine the Hamiltonians let's add H and H' to 40:

$$\left(\mathrm{H} = \mathrm{E_k} + \mathrm{E_p}\right) + \left(H' = E'_k + \mathrm{E_p}\right) \rightarrow H + H' = \mathrm{E_k} + E'_k + 2\mathrm{E_p} \qquad 32.47$$

Applying $T_k = E'_k$ from 27 to 47 we obtain:

$$H + H' = \mathrm{E_k} + E'_k + 2\mathrm{E_p} = \mathrm{E_k} + T_k + 2\mathrm{E_p} \rightarrow H + H' = \mathrm{E_k} + T_k + 2\mathrm{E_p} \qquad 32.48$$

Applying 30 $\mathrm{vp} = \mathrm{E_k} + \mathrm{T_k}$ in 48 we obtain:

$$\mathrm{H} + \mathrm{H'} = \mathrm{E_k} + \mathrm{T_k} + 2\mathrm{E_p} = \mathrm{vp} + 2\mathrm{E_p} \rightarrow \mathrm{H} + \mathrm{H'} = \mathrm{vp} + 2\mathrm{E_p} \qquad 32.49$$

Now defining the Hamiltonians according to 49:

$$H = \frac{1}{2}\mathrm{vp} + \mathrm{E_p} \qquad 32.50$$

$$H' = \frac{1}{2}\mathrm{v'p'} + \mathrm{E_p} \qquad 32.51$$

These Hamiltonians are H = H' invariants.

Adding 50 plus 51 we get:

$$\left(\mathrm{H} = \frac{1}{2}\mathrm{vp} + \mathrm{E_p}\right) + \left(\mathrm{H'} = \frac{1}{2}\mathrm{v'p'} + \mathrm{E_p}\right) \rightarrow \mathrm{H} + \mathrm{H'} = \frac{1}{2}(\mathrm{vp} + \mathrm{v'p'}) + 2\mathrm{E_p} = \mathrm{vp} + 2\mathrm{E_p} \qquad 32.52$$

And we get 52 = 49.

The Hamiltonian $H = \frac{1}{2}\mathrm{vp} + \mathrm{E_p}$ must agree with the Hamiltonian equation $\mathrm{v} = \frac{\partial \mathrm{H}}{\partial \mathrm{p}}$.

$$\frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\left(\frac{1}{2}\mathrm{vp} + \mathrm{E_p}\right) = \frac{\partial}{\partial p}\left(\frac{1}{2}\mathrm{vp}\right) \qquad \frac{\partial \mathrm{E_p}}{\partial p} = zero \qquad 32.53$$

$$\frac{\partial \mathrm{H}}{\partial \mathrm{p}} = \frac{\partial}{\partial \mathrm{p}}\left(\frac{1}{2}\mathrm{vp}\right) = \frac{1}{2}\frac{\partial \mathrm{v}}{\partial \mathrm{p}}\mathrm{p} + \frac{1}{2}\mathrm{v}\frac{\partial \mathrm{p}}{\partial \mathrm{p}} = \frac{1}{2}\frac{\partial \mathrm{v}}{\partial \mathrm{p}}\mathrm{p} + \frac{1}{2}\mathrm{v} \qquad 32.54$$

Deriving the velocity of 10 $\mathrm{v} = \mathrm{c}^2\frac{\mathrm{p}}{\mathrm{E}}$ we have

$$\frac{\partial \mathrm{v}}{\partial \mathrm{p}} = \frac{\partial}{\partial \mathrm{p}}\left(\mathrm{c}^2\frac{\mathrm{p}}{\mathrm{E}}\right) = \mathrm{c}^2\frac{\partial}{\partial \mathrm{p}}(pE^{-1}) = \mathrm{c}^2\frac{\partial \mathrm{p}}{\partial \mathrm{p}}E^{-1} + \mathrm{c}^2 p\frac{\partial(E^{-1})}{\partial \mathrm{p}} = \frac{\mathrm{c}^2}{E} - \mathrm{c}^2 pE^{-1-1=-2}\frac{\partial E}{\partial \mathrm{p}}$$

$$\frac{\partial \mathrm{v}}{\partial \mathrm{p}} = \frac{\mathrm{c}^2}{E} - \mathrm{c}^2 pE^{-1-1=-2}\frac{\partial E}{\partial \mathrm{p}} = \frac{\mathrm{c}^2}{E} - \mathrm{c}^2\frac{\mathrm{p}}{E^2}\frac{\partial E}{\partial \mathrm{p}} \qquad 32.55$$

Now deriving E with respect to p in 18 we get:

$$\frac{\partial}{\partial \mathrm{p}}\left(\frac{\mathrm{E}^2}{\mathrm{c}^2} = \mathrm{p}^2 + \mathrm{m}_0^2\mathrm{c}^2\right) \rightarrow \frac{2\mathrm{E}}{\mathrm{c}^2}\frac{\partial \mathrm{E}}{\partial \mathrm{p}} = 2\mathrm{p} \rightarrow \frac{\mathrm{E}}{\mathrm{c}^2}\frac{\partial \mathrm{E}}{\partial \mathrm{p}} = \mathrm{p} \rightarrow \frac{\partial \mathrm{E}}{\partial \mathrm{p}} = \mathrm{c}^2\frac{\mathrm{P}}{\mathrm{E}} = \mathrm{v} \qquad 32.56$$

Applying 55 and 56 out of 54 we obtain:

$$\frac{\partial \mathrm{H}}{\partial \mathrm{p}} = \frac{1}{2}\frac{\partial \mathrm{v}}{\partial \mathrm{p}}\mathrm{p} + \frac{1}{2}\mathrm{v} = \frac{1}{2}\left(\frac{\mathrm{c}^2}{\mathrm{E}} - \mathrm{c}^2\frac{\mathrm{p}}{\mathrm{E}^2}\frac{\partial \mathrm{E}}{\partial \mathrm{p}}\right)\mathrm{p} + \frac{1}{2}\mathrm{v} = \frac{1}{2}\left[\frac{\mathrm{c}^2}{\mathrm{E}} - \mathrm{c}^2\frac{\mathrm{p}}{\mathrm{E}^2}(\mathrm{v})\right]\mathrm{p} + \frac{1}{2}\mathrm{v}$$

$$\frac{\partial \mathrm{H}}{\partial \mathrm{p}} = \frac{1}{2}\left[\frac{\mathrm{c}^2}{\mathrm{E}} - \mathrm{c}^2\frac{\mathrm{p}}{\mathrm{E}^2}(\mathrm{v})\right]\mathrm{p} + \frac{1}{2}\mathrm{v} = \frac{1}{2}\frac{\mathrm{c}^2}{\mathrm{E}}\mathrm{p} - \frac{1}{2}\mathrm{c}^2\frac{\mathrm{p}^2}{\mathrm{E}^2}\mathrm{v} + \frac{1}{2}\mathrm{v} = \frac{1}{2}\mathrm{v} - \frac{1}{2}\frac{\mathrm{v}^2}{\mathrm{c}^2}\mathrm{v} + \frac{1}{2}\mathrm{v}$$

$$\frac{\partial H}{\partial p} = \frac{1}{2}v - \frac{1}{2}\frac{v^2}{c^2}v + \frac{1}{2}v = v - \frac{1}{2}\frac{v^2}{c^2}v = v\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)$$

$$\frac{\partial H}{\partial p} = v\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) = v \quad 32.57$$

In 57 we consider the term $\frac{1}{2}\frac{v^2}{c^2} = zero$ or we could consider that the speed of light has the character of being infinite in Quantum Mechanics (QM) $\frac{1}{2}\frac{v^2}{(c=\infty)^2} = zero$.

Applying the formula 10 of the velocity $v = c^2\frac{p}{E}$ to the Hamiltonian of 50 we obtain:

$$H = \frac{1}{2}vp + E_p = \frac{1}{2}\left(c^2\frac{p}{E}\right)p + E_p = \frac{1}{2}\frac{c^2p^2}{E} + E_p \rightarrow H = \frac{1}{2}\frac{c^2p^2}{E} + E_p \quad 32.58$$

And at 58 we have the ambivalent kinetic energy of 11 $E_k = \frac{1}{2}\frac{c^2p^2}{E}$.

From 58 we get the value of p:

$$p = \sqrt{\frac{2E}{c^2}\left(H - E_p\right)} = \sqrt{\frac{2}{c^2}\frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}\left(H - E_p\right)} = \sqrt{\frac{2m_o}{\sqrt{1-\frac{v^2}{c^2}}}\left(H - E_p\right)} \quad 32.59$$

In this doing $c = \infty$ we obtain:

$$p = \sqrt{\frac{2m_o}{\sqrt{1-\frac{v^2}{c^2}}}\left(H - E_p\right)} = \sqrt{\frac{2m_o}{\sqrt{1-\frac{v^2}{(\infty)^2}}}\left(H - E_p\right)} = \sqrt{2m_o\left(H - E_p\right)} \rightarrow p = \sqrt{2m_o\left(H - E_p\right)} \quad 32.60$$

In 60 we have the p value of the theory of Erwin Schödinger.

Applying 60 out of 10 we get the particle velocity:

$$v = c^2\frac{p}{E} = \frac{c^2}{E}\sqrt{\frac{2E}{c^2}\left(H - E_p\right)} \rightarrow v = \sqrt{\frac{2c^2}{E}\left(H - E_p\right)} \quad 32.61$$

In what follows the development is approximately the method of the own Erwin Schödinger.

In Hamiltonian 58 applying the Hamilton Jacobi equations to the x axis we obtain:

$$\frac{\partial S}{\partial q} = \frac{\partial S}{\partial x} = p \qquad -\frac{\partial S}{\partial t} = H \qquad \frac{\partial S}{\partial t} = -H\left(q, \frac{\partial S}{\partial q}, t\right) = -H\left(x, \frac{\partial S}{\partial x}, t\right) \quad 32.62$$

$$H = \frac{1}{2}\frac{c^2p^2}{E} + E_p = \frac{1}{2}\frac{c^2}{E}\left(\frac{\partial S}{\partial x}\right)^2 + E_p = -\frac{\partial S}{\partial t} \quad 32.63$$

$$\frac{1}{2}\frac{c^2}{E}\left(\frac{\partial S}{\partial x}\right)^2 + E_p + \frac{\partial S}{\partial t} = zero \quad 32.64$$

For a conservative system the Hamilton Jacobi equations are given by:

$$\frac{\partial S}{\partial x} = p \qquad \frac{\partial S}{\partial t} = -H \quad 32.65$$

From 65 it is concluded that the action S can be in the form:

$$S = S(x,t) = f(x) + g(t) = constante \quad 32.66$$

Where f = f(x) is a function of x that should result in $\frac{\partial S}{\partial x} = \frac{\partial f}{\partial x} = p$ 32.67

Applying $\frac{\partial S}{\partial x} = \frac{\partial f}{\partial x}$ e $\frac{\partial S}{\partial t} = -H$ in 64 we obtain:

$$\frac{1}{2}\frac{c^2}{E}\left(\frac{\partial S}{\partial x}\right)^2 + E_p + \frac{\partial S}{\partial t} = zero \rightarrow \frac{1}{2}\frac{c^2}{E}\left(\frac{\partial f}{\partial x}\right)^2 + E_p - H = zero \quad 32.68$$

Now let's make the transformation in 68 $f = f(x) = kln\Psi$ 32.69

Where k is a constant.

The f function of 69 has an analogy with entropy.

Applying 69 out of 68 we obtain:

$$\frac{1}{2}\frac{c^2}{E}\left(\frac{\partial f}{\partial x}\right)^2 + E_p - H = zero \rightarrow \frac{1}{2}\frac{c^2}{E}\left[\frac{\partial(kln\Psi)}{\partial x}\right]^2 + E_p - H = zero \quad 32.70$$

$$\frac{1}{2}\frac{c^2}{E}\left[\frac{\partial(kln\Psi)}{\partial x}\right]^2 + E_p - H = zero \rightarrow \frac{1}{2}\frac{c^2}{E}\left(\frac{k}{\Psi}\frac{\partial\Psi}{\partial x}\right)^2 + E_p - H = zero$$

$$\frac{1}{2}\frac{c^2}{E}\left(\frac{k}{\Psi}\frac{\partial\Psi}{\partial x}\right)^2 + E_p - H = zero \rightarrow \frac{1}{2}\frac{c^2k^2}{E}\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(E_p - H\right)\Psi^2 = zero$$

$$\frac{1}{2}\frac{c^2k^2}{E}\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(E_p - H\right)\Psi^2 = zero \quad 32.71$$

Now suppose 71 is not null and has a remainder R in the form:

$$R = R\left(\Psi, \frac{\partial\Psi}{\partial x}, x\right) = \frac{1}{2}\frac{c^2k^2}{E}\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(E_p - H\right)\Psi^2 \quad 32.72$$

The rest R must be a minimum so it must meet the functional:

$$\int_{-\infty}^{+\infty} R\left(\Psi, \frac{\partial\Psi}{\partial x}, x\right) dx \quad 32.73$$

And we get from the Euler Lagrange equation of the functional:

$$\frac{\partial R}{\partial\Psi} - \frac{\partial}{\partial x}\left[\frac{\partial R}{\partial\left(\frac{\partial\Psi}{\partial x}\right)}\right] = zero \rightarrow \frac{\partial}{\partial\Psi}\left[\frac{1}{2}\frac{c^2k^2}{E}\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(E_p - H\right)\Psi^2\right] - \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial\left(\frac{\partial\Psi}{\partial x}\right)}\left[\frac{1}{2}\frac{c^2k^2}{E}\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(E_p - H\right)\Psi^2\right]\right\} = zero$$

$$\frac{\partial}{\partial\Psi}\left[\frac{1}{2}\frac{c^2k^2}{E}\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(E_p - H\right)\Psi^2\right] - \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial\left(\frac{\partial\Psi}{\partial x}\right)}\left[\frac{1}{2}\frac{c^2k^2}{E}\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(E_p - H\right)\Psi^2\right]\right\} = 2\left(E_p - H\right)\Psi - \frac{\partial}{\partial x}\left[\frac{1}{2}\frac{c^2k^2}{E}2\left(\frac{\partial\Psi}{\partial x}\right)\right] = zero$$

$$2\left(E_p - H\right)\Psi - \frac{\partial}{\partial x}\left[\frac{1}{2}\frac{c^2k^2}{E}2\left(\frac{\partial\Psi}{\partial x}\right)\right] = \left(E_p - H\right)\Psi - \frac{1}{2}\frac{c^2k^2}{E}\frac{\partial^2\Psi}{\partial x^2} = -\frac{1}{2}\frac{c^2k^2}{E}\frac{\partial^2\Psi}{\partial x^2} + \left(E_p - H\right)\Psi = zero$$

$$-\frac{1}{2}\frac{c^2k^2}{E}\frac{\partial^2\Psi}{\partial x^2} + \left(E_p - H\right)\Psi = zero \rightarrow -\frac{1}{2}\frac{c^2k^2}{E}\frac{\partial^2\Psi}{\partial x^2} + E_p\Psi = H\Psi$$

$$-\frac{1}{2}\frac{c^2k^2}{E}\frac{\partial^2\Psi}{\partial x^2} + E_p\Psi = H\Psi \quad 32.74$$

In 74 we have $\Psi = \Psi(x) \rightarrow \frac{\partial^2\Psi}{\partial x^2} = \frac{d^2\Psi}{dx^2}$ 32.75

$$-\frac{1}{2}\frac{c^2k^2}{E}\frac{\partial^2\Psi}{\partial x^2} + E_p\Psi = H\Psi \rightarrow -\frac{1}{2}\frac{c^2k^2}{E}\frac{d^2\Psi}{dx^2} + E_p\Psi = H\Psi \quad 32.76$$

Applying the energy $E = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}$ in 76 we obtain:

$$-\frac{1}{2}\frac{c^2k^2}{E}\frac{d^2\Psi}{dx^2} + E_p\Psi = H\Psi \rightarrow -\frac{1}{2}\frac{c^2k^2}{\frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}}}\frac{d^2\Psi}{dx^2} + E_p\Psi = H\Psi \rightarrow -\frac{1}{2}\frac{k^2}{\frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}}\frac{d^2\Psi}{dx^2} + E_p\Psi = H\Psi \quad 32.77$$

Making in 77 $c = \infty$ we obtain:

$$-\frac{1}{2}\frac{k^2}{\frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}}\frac{d^2\Psi}{dx^2} + E_p\Psi = H\Psi \rightarrow -\frac{1}{2}\frac{k^2}{\frac{m_o}{\sqrt{1-\frac{v^2}{(c=\infty)^2}}}}\frac{d^2\Psi}{dx^2} + E_p\Psi = H\Psi \rightarrow -\frac{1}{2}\frac{k^2}{m_o}\frac{d^2\Psi}{dx^2} + E_p\Psi = H\Psi \quad 32.78$$

Now in 78 making k equal to Planck's constant $k = \hbar$ and replacing H with E because now it doesn't cause any more confusion:

$$-\frac{1}{2}\frac{k^2}{\mathrm{m_o}}\frac{d^2\Psi}{dx^2} + \mathrm{E_p}\Psi = \mathrm{H}\Psi \rightarrow -\frac{1}{2}\frac{\hbar^2}{\mathrm{m_o}}\frac{d^2\Psi}{dx^2} + \mathrm{E_p}\Psi = \mathrm{E}\Psi$$

$$-\frac{\hbar^2}{2\mathrm{m_o}}\frac{\mathrm{d}^2\Psi}{\mathrm{dx}^2} + \mathrm{E_p}\Psi = \mathrm{E}\Psi \qquad 32.79$$

And we have 79 equal to Erwin Schödinger. equation 31.49. The method used in this work is approximately the one used by Erwin Schödinger.

§33 Hyperbolic Relativistic Energy

Next, we will conclude that relativistic energy is a hyperbolic function.

From the energy of 30.9 we obtain:

$$\mathrm{E} = \mathrm{c}\sqrt{\mathrm{p}^2 + \mathrm{m}_0^2\mathrm{c}^2} \qquad 33.1$$

$$\mathrm{E}^2 = \mathrm{c}^2\mathrm{p}^2 + \mathrm{m}_0^2\mathrm{c}^4 \rightarrow \mathrm{E}^2 - \mathrm{c}^2\mathrm{p}^2 = \mathrm{m}_0^2\mathrm{c}^4 \qquad 33.2$$

$$(\mathrm{E} + \mathrm{cp})(\mathrm{E} - \mathrm{cp}) = \mathrm{m_0c^2}.\mathrm{m_0c^2} \qquad 33.3$$

$$\frac{(\mathrm{E}+\mathrm{cp})}{\mathrm{m_0c^2}}.\frac{(\mathrm{E}-\mathrm{cp})}{\mathrm{m_0c^2}} = 1 \qquad 33.4$$

In this denominating:

$$\mathrm{e}^{\emptyset} = \frac{(\mathrm{E}+\mathrm{cp})}{\mathrm{m_0c^2}} = \frac{\left(\frac{\mathrm{m_0c^2}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} + \frac{\mathrm{cm_0v}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}}\right)}{\mathrm{m_0c^2}} = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} + \frac{\frac{\mathrm{v}}{\mathrm{c}}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} = \frac{1+\frac{\mathrm{v}}{\mathrm{c}}}{\sqrt{\left(1-\frac{\mathrm{v}}{\mathrm{c}}\right)\left(1+\frac{\mathrm{v}}{\mathrm{c}}\right)}} = \sqrt{\frac{\left(1+\frac{\mathrm{v}}{\mathrm{c}}\right)}{\left(1-\frac{\mathrm{v}}{\mathrm{c}}\right)}} \qquad 33.5$$

$$\emptyset = \ln\left[\sqrt{\frac{\left(1+\frac{\mathrm{v}}{\mathrm{c}}\right)}{\left(1-\frac{\mathrm{v}}{\mathrm{c}}\right)}}\right] \qquad 33.6$$

$$\mathrm{e}^{-\emptyset} = \frac{(\mathrm{E}-\mathrm{cp})}{\mathrm{m_0c^2}} = \frac{\left(\frac{\mathrm{m_0c^2}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} - \frac{\mathrm{cm_0v}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}}\right)}{\mathrm{m_0c^2}} = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} - \frac{\frac{\mathrm{v}}{\mathrm{c}}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} = \frac{1-\frac{\mathrm{v}}{\mathrm{c}}}{\sqrt{\left(1-\frac{\mathrm{v}}{\mathrm{c}}\right)\left(1+\frac{\mathrm{v}}{\mathrm{c}}\right)}} = \sqrt{\frac{\left(1-\frac{\mathrm{v}}{\mathrm{c}}\right)}{\left(1+\frac{\mathrm{v}}{\mathrm{c}}\right)}} \qquad 33.7$$

$$-\emptyset = \ln\left[\sqrt{\frac{\left(1-\frac{\mathrm{v}}{\mathrm{c}}\right)}{\left(1+\frac{\mathrm{v}}{\mathrm{c}}\right)}}\right] \qquad 33.8$$

$\mathrm{e}^{\emptyset}.\mathrm{e}^{-\emptyset} = 1$ that is in agreement with 4. 33.9

Now calling the hyperbolic cosine (ch) as:

$$\mathrm{x} = \mathrm{ch}\emptyset = \frac{\mathrm{e}^{\emptyset}+\mathrm{e}^{-\emptyset}}{2} = \frac{1}{2}\left[\frac{(\mathrm{E}+\mathrm{cp})}{\mathrm{m_0c^2}} + \frac{(\mathrm{E}-\mathrm{cp})}{\mathrm{m_0c^2}}\right] = \frac{\mathrm{E}}{\mathrm{m_0c^2}} = \frac{1}{\mathrm{m_0c^2}}\frac{\mathrm{m_0c^2}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} = \frac{1}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} \qquad 33.10$$

And naming the hyperbolic sine (sh) as:

$$\mathrm{y} = \mathrm{sh}\emptyset = \frac{\mathrm{e}^{\emptyset}-\mathrm{e}^{-\emptyset}}{2} = \frac{1}{2}\left[\frac{(\mathrm{E}+\mathrm{cp})}{\mathrm{m_0c^2}} - \frac{(\mathrm{E}-\mathrm{cp})}{\mathrm{m_0c^2}}\right] = \frac{\mathrm{cp}}{\mathrm{m_0c^2}} = \frac{1}{\mathrm{m_0c^2}}\frac{\mathrm{cm_0v}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} = \frac{\frac{\mathrm{v}}{\mathrm{c}}}{\sqrt{1-\frac{\mathrm{v}^2}{\mathrm{c}^2}}} \qquad 33.11$$

To prove that cosine and sine agree with the hyperbola equation let's make cosine equal to x and sine equal to y of the hyperbola equation 12:

$$\mathrm{x}^2 - \mathrm{y}^2 = 1 \qquad 33.12$$

$$\left(\frac{E}{m_0c^2}\right)^2 - \left(\frac{cp}{m_0c^2}\right)^2 = \frac{E^2-c^2p^2}{m_0^2c^4} = \frac{m_0^2c^4}{m_0^2c^4} = 1 \qquad 33.13$$

$$\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 - \left(\frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 = \frac{1}{1-\frac{v^2}{c^2}} - \frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} = \frac{1-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} = 1 \qquad 33.14$$

Since $e^{\emptyset}$ and $e^{-\emptyset}$ are always positive, the hyperbolic cosine is always greater than zero:

$$x = ch\emptyset = \frac{e^{\emptyset}+e^{-\emptyset}}{2} = \frac{E}{m_0c^2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} > zero \qquad 33.15$$

So there is no negative energy.

Now defining the hyperbolic tangent, secant cotangent and cosecant have:

$$th\emptyset = \frac{sh\emptyset}{ch\emptyset} = \frac{e^{\emptyset}-e^{-\emptyset}}{e^{\emptyset}+e^{-\emptyset}} = \frac{\frac{cp}{m_0c^2}}{\frac{E}{m_0c^2}} = \frac{cp}{E} = \frac{\frac{cm_0v}{\sqrt{1-\frac{v^2}{c^2}}}}{\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}} = \frac{v}{c} \qquad 33.16$$

$$coth\emptyset = \frac{ch\emptyset}{sh\emptyset} = \frac{e^{\emptyset}+e^{-\emptyset}}{e^{\emptyset}-e^{-\emptyset}} = \frac{\frac{E}{m_0c^2}}{\frac{cp}{m_0c^2}} = \frac{E}{cp} = \frac{\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}}{\frac{cm_0v}{\sqrt{1-\frac{v^2}{c^2}}}} = \frac{c}{v} \qquad 33.17$$

$$sech\emptyset = \frac{1}{ch\emptyset} = \frac{2}{e^{\emptyset}+e^{-\emptyset}} = \frac{1}{\frac{E}{m_0c^2}} = \frac{m_0c^2}{E} = \frac{m_0c^2}{\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}} = \sqrt{1-\frac{v^2}{c^2}} \qquad 33.18$$

$$cossech\emptyset = \frac{1}{sh\emptyset} = \frac{2}{e^{\emptyset}-e^{-\emptyset}} = \frac{1}{\frac{cp}{m_0c^2}} = \frac{m_0c^2}{cp} = \frac{m_0c^2}{\frac{cm_0v}{\sqrt{1-\frac{v^2}{c^2}}}} = \frac{c}{v}\sqrt{1-\frac{v^2}{c^2}} \qquad 33.19$$

Trigonometric Functions ⇆ Hyperbolic Functions

Construction of relationships that transform hyperbolic functions into trigonometric functions.

The Pythagorean formula for a right triangle with hypotenuse “h” and side “a” adjacent to angle α and side “b” opposite angle α is:

$$h^2 = a^2 + b^2 \qquad 33.20$$

For this triangle we have the following trigonometric functions $ft = ft(\alpha)$ with angle α:

$$a = h.cos\alpha \qquad b = h.sen\alpha \qquad 33.21$$

Reshaping the Pythagorean formula gives:

$$h^2 = a^2 + b^2 \rightarrow b^2 = h^2 - a^2 = (h+a)(h-a) \rightarrow \left(\frac{h+a}{b}\right)\left(\frac{h-a}{b}\right) = e^{\emptyset}e^{-\emptyset} = 1 \qquad 33.22$$

This is divided into the following hyperbolic functions $fh = fh(\emptyset)$ with angle $\emptyset$:

$$e^{\emptyset} = \frac{h+a}{b} > zero \qquad 33.23$$

$$e^{-\emptyset} = \frac{h-a}{b} > zero \qquad 33.24$$

Where applying the trigonometric functions we obtain $fh(\emptyset) = ft(\alpha)$:

$e^{\emptyset} = \frac{h+a}{b} = \frac{h+h.cos\alpha}{h.sen\alpha} = \frac{1+cos\alpha}{sen\alpha}$ 33.25

$e^{-\emptyset} = \frac{h-a}{b} = \frac{h-h.cos\alpha}{h.sen\alpha} = \frac{1-cos\alpha}{sen\alpha}$ 33.26

The real equality of the functions $fh(\emptyset) = ft(\alpha)$ only occurs if the angle of the hyperbolic function is equal to the angle of the trigonometric function, that is, if $fh(\emptyset) = ft(\emptyset)$ where both are hyperbolic functions or $fh(\alpha) = ft(\alpha)$

where both are trigonometric functions.

From trigonometry we have:

$tg\left(\frac{\alpha}{2}\right) = \frac{1-cos\alpha}{sen\alpha} = \frac{sen\alpha}{1+cos\alpha} = \sqrt{\frac{1-cos\alpha}{1+cos\alpha}}$ 33.27

Applying 27 we obtain the fundamental function of the trigonometric angle α as a function of the hyperbolic angle $\emptyset$, $\alpha = \alpha(\emptyset)$:

$e^{\emptyset} = \frac{1+cos\alpha}{sen\alpha} = \frac{1}{tg\left(\frac{\alpha}{2}\right)} = \frac{1}{\sqrt{\frac{1-cos\alpha}{1+cos\alpha}}} = \sqrt{\frac{1+cos\alpha}{1-cos\alpha}}$ 33.28

$e^{-\emptyset} = \frac{1-cos\alpha}{sen\alpha} = tg\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-cos\alpha}{1+cos\alpha}}$ 33.29

$\alpha = 2arctg\left(e^{-\emptyset}\right)$ 33.30

The substitution of the angle 30 $\alpha = \alpha(\emptyset)$ in the trigonometric functions transforms it into hyperbolic functions with the necessary restrictions of existence, in the following form:

$fh(\emptyset) = ft(\alpha) = ft[\alpha(\emptyset)] = ft(\emptyset) \rightarrow fh(\emptyset) = ft(\emptyset)$ 33.31

From 28 and 29 we obtain the fundamental formulas of the hyperbolic angle $\emptyset$ as a function of the trigonometric angle α, $\emptyset = \emptyset(\alpha)$:

$ln\left(e^{\emptyset}\right) = ln\left[\frac{1}{tg\left(\frac{\alpha}{2}\right)}\right] \rightarrow \emptyset = \emptyset(\alpha) = ln\left[\frac{1}{tg\left(\frac{\alpha}{2}\right)}\right]$ 33.32

$ln\left(e^{-\emptyset}\right) = ln\left[tg\left(\frac{\alpha}{2}\right)\right] \rightarrow -\emptyset = ln\left[tg\left(\frac{\alpha}{2}\right)\right] \rightarrow \emptyset = \emptyset(\alpha) = ln\left[\frac{1}{tg\left(\frac{\alpha}{2}\right)}\right]$ 33.33

The functions (30) $\alpha = \alpha(\emptyset)$ and (32) $\emptyset = \emptyset(\alpha)$ are inverses of each other.

The substitution of angle 32 $\emptyset = \emptyset(\alpha)$ in the hyperbolic functions transforms it into trigonometric functions with the appropriate existence restrictions, in the following form:

$ft(\alpha) = fh(\emptyset) = fh\,[\emptyset(\alpha)] = fh(\alpha) \rightarrow ft(\alpha) = fh(\alpha)$ 33.34

In the unitary hyperbola $\mathrm{x}^2 - y^2 = 1$ applying the functions $\mathrm{x} = \mathrm{ch}\emptyset$ and $\mathrm{y} = \mathrm{sh}\emptyset$ we get:

$\mathrm{x}^2 - y^2 = \mathrm{ch}^2\emptyset - \mathrm{sh}^2\emptyset = (\mathrm{ch}\emptyset + \mathrm{sh}\emptyset)(\mathrm{ch}\emptyset - \mathrm{sh}\emptyset) = e^{\emptyset}.e^{-\emptyset} = 1$ 33.35

Breaking it down into two functions yields the hyperbolic cosine "$\mathrm{ch}\emptyset$" and hyperbolic sine "$\mathrm{sh}\emptyset$" functions:

$\mathrm{ch}\emptyset + \mathrm{sh}\emptyset = e^{\emptyset} \rightarrow x = \mathrm{ch}\emptyset = \frac{e^{\emptyset}+e^{-\emptyset}}{2}$ 33.36

$\mathrm{ch}\emptyset - \mathrm{sh}\emptyset = e^{-\emptyset} \rightarrow y = \mathrm{sh}\emptyset = \frac{e^{\emptyset}-e^{-\emptyset}}{2}$ 33.37

In 36 and 37 we have the fundamental properties of the hyperbolic functions.

Applying to the hyperbolic cosine $\mathrm{ch}\emptyset$, the previous variables are obtained:

$$x = ch\emptyset = \frac{e^{\emptyset}+e^{-\emptyset}}{2} = \frac{1}{2}\left(\frac{h+a}{b} + \frac{h-a}{b}\right) = \frac{h}{b} = \frac{h}{h.sen\alpha} = \frac{1}{sen\alpha} = cosec\alpha \quad 33.38$$

Applying to the hyperbolic sine $\mathrm{sh}\emptyset$, the previous variables are obtained:

$$y = sh\emptyset = \frac{e^{\emptyset}-e^{-\emptyset}}{2} = \frac{1}{2}\left(\frac{h+a}{b} - \frac{h-a}{b}\right) = \frac{a}{b} = \frac{h.cos\alpha}{h.sen\alpha} = \frac{cos\alpha}{sen\alpha} = cotg\alpha \quad 33.39$$

Applying the hyperbolic cosine $x = \mathrm{ch}\emptyset = cosec\alpha$ and the hyperbolic sine $y = \mathrm{sh}\emptyset = cotg\alpha$ to the unitary hyperbola equation $x^2 - y^2 = 1$ we get:

$$x^2 - y^2 = ch^2\emptyset - sh^2\emptyset = cosec^2\alpha - cotg^2\alpha = 1 \quad 33.40$$

Which is a result of trigonometry.

With the relations of the hyperbolic cosine $\mathrm{ch}\emptyset$ and the hyperbolic sine $\mathrm{sh}\emptyset$ we can define the other relations between the trigonometric functions and the hyperbolic functions:

$$tgh\emptyset = \frac{sh\emptyset}{ch\emptyset} = \frac{\frac{cos\alpha}{sen\alpha}}{\frac{1}{sen\alpha}} = cos\alpha \quad 33.41$$

$$cotgh\emptyset = \frac{ch\emptyset}{sh\emptyset} = \frac{\frac{1}{sen\alpha}}{\frac{cos\alpha}{sen\alpha}} = \frac{1}{cos\alpha} = sec\alpha \quad 33.42$$

$$sech\emptyset = \frac{1}{ch\emptyset} = \frac{1}{\frac{1}{sen\alpha}} = sen\alpha \quad 33.43$$

$$cosech\emptyset = \frac{1}{sh\emptyset} = \frac{1}{\frac{cos\alpha}{sen\alpha}} = \frac{sen\alpha}{cos\alpha} = tg\alpha \quad 33.44$$

$$sech^2\emptyset + tgh^2\emptyset = sen^2\alpha + cos^2\alpha = 1 \quad 33.45$$

$$cotgh^2\emptyset - cosech^2\emptyset = sec^2\alpha - tg^2\alpha = 1 \quad 33.46$$

Construction of the already known relationships that transform the hyperbolic functions into the exponential form of a complex number.

Next, we will use Euler's formulas:

$$e^{i\alpha} = cos\alpha + isen\alpha \qquad e^{-i\alpha} = cos\alpha - isen\alpha \quad 33.47$$

Reshaping the Pythagorean formula, we get:

$$h^2 = a^2 + b^2 = a^2 - (ib)^2 = (a+ib)(a-ib) \rightarrow \frac{(a+ib)}{h}\frac{(a-ib)}{h} = e^{\emptyset}e^{-\emptyset} = 1 \quad 33.48$$

This breaks down into the following complex hyperbolic functions:

$$e^{\emptyset} = \frac{a+ib}{h} > zero \quad 33.49$$

$$e^{-\emptyset} = \frac{a-ib}{h} > zero \quad 33.50$$

For this triangle we have the trigonometric relations:

$$\frac{a}{h} = cos\alpha \qquad \frac{b}{h} = sen\alpha \quad 33.51$$

Applying trigonometric relations, we get:

$$e^{\emptyset} = \frac{a+ib}{h} = \frac{a}{h} + i\frac{b}{h} = cos\alpha + isen\alpha \quad 33.52$$

$$e^{-\emptyset} = \frac{a-ib}{h} = \frac{a}{h} - i\frac{b}{h} = cos\alpha - isen\alpha \quad 33.53$$

To conform to Euler's formulas we must change the hyperbolic arguments to $\emptyset = i\alpha$ and thus we obtain the hyperbolic functions written as the exponential form of a complex number:

$e^{\emptyset} = e^{i\alpha} = cos\alpha + isen\alpha$ 33.54

$e^{-\emptyset} = e^{-i\alpha} = cos\alpha - isen\alpha$ 33.55

Calling the cosseno $chi\alpha$ hyperbolic complex as:

$x = chi\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2} = \frac{1}{2}[(cos\alpha + isen\alpha) + (cos\alpha - isen\alpha)] = cos\alpha$ 33.56

And naming the sine $shi\alpha$ hyperbolic complex as:

$y = shi\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2} = \frac{1}{2}[(cos\alpha + isen\alpha) - (cos\alpha - isen\alpha)] = isen\alpha$ 33.57

Applying the cosine $x = chi\alpha = cos\alpha$ hyperbolic complex and the sine $y = shi\alpha = isen\alpha$ hyperbolic complex in the equation of the unit hyperbola $x^2 - y^2 = 1$ results:

$x^2 - y^2 = ch^2 i\alpha - sh^2 i\alpha = cos^2\alpha - i^2 sen^2\alpha = cos^2\alpha + sen^2\alpha = 1$ 33.58

Which is a result of trigonometry.

With the relationships of the hyperbolic cosine $chi\alpha = cos\alpha$ and the hyperbolic sine $shi\alpha = isen\alpha$ we can define the other relationships between complex trigonometric functions and complex hyperbolic functions.

Construction of relationships that transform hyperbolic functions into trigonometric functions similar to those that occur in Gudermannian functions.

The Pythagorean formula for a right triangle with hypotenuse “h” and side “a” adjacent to angle α and side “b” opposite angle α is:

$h^2 = a^2 + b^2$ 33.59

For this triangle we have the trigonometric relations:

$a = h.cos\alpha$ $b = h.sen\alpha$ 33.60

Reshaping the Pythagorean formula gives:

$h^2 = a^2 + b^2 \rightarrow a^2 = h^2 - b^2 = (h+b)(h-b) \rightarrow \left(\frac{h+b}{a}\right)\left(\frac{h-b}{a}\right) = e^{\beta}.e^{-\beta} = 1$ 33.61

This is divided into the following hyperbolic functions:

$e^{\beta} = \frac{h+b}{a} > zero$ 33.62

$e^{-\beta} = \frac{h-b}{a} > zero$ 33.63

Where applying the trigonometric relations we obtain:

$e^{\beta} = \frac{h+b}{a} = \frac{h+h.sen\alpha}{h.cos\alpha} = \frac{1+sen\alpha}{cos\alpha}$ 33.64

$e^{-\beta} = \frac{h-b}{a} = \frac{h-hsen\alpha}{h.cos\alpha} = \frac{1-sen\alpha}{cos\alpha}$ 33.65

From these we obtain the fundamental formulas of the hyperbolic angle β:

$ln(e^{\beta}) = ln\left(\frac{1+sen\alpha}{cos\alpha}\right) \rightarrow \beta = ln\left(\frac{1+sen\alpha}{cos\alpha}\right)$ 33.66

$ln(e^{-\beta}) = ln\left(\frac{1-sen\alpha}{cos\alpha}\right) \rightarrow \beta = -ln\left(\frac{1-sen\alpha}{cos\alpha}\right)$ 33.67

Denominating the hyperbolic cosine $ch\beta$ as:

$x = ch\beta = \frac{e^{\beta}+e^{-\beta}}{2} = \frac{1}{2}\left(\frac{h+b}{a} + \frac{h-b}{a}\right) = \frac{h}{a} = \frac{h}{h.cos\alpha} = \frac{1}{cos\alpha} = sec\alpha$ 33.68

And calling the hyperbolic sine $sh\beta$ as:

$$y = sh\beta = \frac{e^{\beta}-e^{-\beta}}{2} = \frac{1}{2}\left(\frac{h+b}{a} - \frac{h-b}{a}\right) = \frac{b}{a} = \frac{h.sen\alpha}{h.cos\alpha} = \frac{sen\alpha}{cos\alpha} = tg\alpha \qquad 33.69$$

Applying the hyperbolic cosine $x = ch\beta = sec\alpha$ and the hyperbolic sine $y = sh\beta = tg\alpha$ to the unitary hyperbola equation $x^2 - y^2 = 1$ we get:

$$x^2 - y^2 = ch^2\beta - sh^2\beta = sec^2\alpha - tg^2\alpha = 1 \qquad 33.70$$

Which is a result of trigonometry.

With the relations of the hyperbolic cosine $ch\beta$ and the hyperbolic sine $sh\beta$ we can define the other relations between the trigonometric functions and the hyperbolic functions:

$$tgh\beta = \frac{sh\beta}{ch\beta} = \frac{\frac{sen\alpha}{cos\alpha}}{\frac{1}{cos\alpha}} = sen\alpha \qquad 33.71$$

$$cotgh\beta = \frac{ch\beta}{sh\beta} = \frac{\frac{1}{cos\alpha}}{\frac{sen\alpha}{cos\alpha}} = \frac{1}{sen\alpha} = cosec\alpha \qquad 33.72$$

$$sech\beta = \frac{1}{ch\beta} = \frac{1}{\frac{1}{cos\alpha}} = cos\alpha \qquad 33.73$$

$$cosech\beta = \frac{1}{sh\beta} = \frac{1}{\frac{sen\alpha}{cos\alpha}} = \frac{cos\alpha}{sen\alpha} = cotg\alpha \qquad 33.74$$

$$sech^2\beta + tgh^2\beta = cos^2\alpha + sen^2\alpha = 1 \qquad 33.75$$

$$cotgh^2\beta - cosech^2\beta = cosec^2\alpha - cotg^2\alpha = 1 \qquad 33.76$$

§34 Hyperbolic equations similar to Paul Adrien Maurice Dirac's equations

In what follows we are always dealing with the free particle $\mathrm{E}_p = zero$.

Writing the relativistic energy equation 30.9:

$$\mathrm{E} = \mathrm{c}\sqrt{\mathrm{m}_0^2\mathrm{c}^2 + \mathrm{p}^2} \to \mathrm{E}^2 = \mathrm{c}^2\mathrm{p}^2 + \mathrm{m}_0^2\mathrm{c}^4 \quad 34.1$$

$$\mathrm{E}^2 = \mathrm{c}^2\mathrm{p}_x^2 + \mathrm{c}^2\mathrm{p}_y^2 + \mathrm{c}^2\mathrm{p}_z^2 + \mathrm{m}_0^2\mathrm{c}^4 \qquad \mathrm{p}^2 = \mathrm{p}_x^2 + \mathrm{p}_y^2 + \mathrm{p}_z^2 \quad 34.2$$

$$\frac{\mathrm{E}^2}{\mathrm{c}^2} = \mathrm{p}_x^2 + \mathrm{p}_y^2 + \mathrm{p}_z^2 + \mathrm{m}_0^2\mathrm{c}^2 \quad 34.3$$

Dirac proposed that the product of the two following equations results in 3.

$$\frac{\mathrm{E}}{\mathrm{c}} = \alpha_1\mathrm{p}_x + \alpha_2\mathrm{p}_y + \alpha_3\mathrm{p}_z + \alpha_4\mathrm{m}_0\mathrm{c} \quad 34.4$$

$$\frac{\mathrm{E}}{\mathrm{c}} = \alpha_1\mathrm{p}_x + \alpha_2\mathrm{p}_y + \alpha_3\mathrm{p}_z + \alpha_4\mathrm{m}_0\mathrm{c} \quad 34.5$$

Making the 4x5 product we get:

$$\frac{\mathrm{E}^2}{\mathrm{c}^2} = \alpha_1\alpha_1\mathrm{p}_x\mathrm{p}_x + \alpha_2\alpha_1\mathrm{p}_x\mathrm{p}_y + \alpha_3\alpha_1\mathrm{p}_x\mathrm{p}_z + \mathrm{m}_0\mathrm{c}\alpha_4\alpha_1\mathrm{p}_x + \alpha_1\alpha_2\mathrm{p}_x\mathrm{p}_y + \alpha_2\alpha_2\mathrm{p}_y\mathrm{p}_y + \alpha_3\alpha_2\mathrm{p}_y\mathrm{p}_z + \mathrm{m}_0\mathrm{c}\alpha_4\alpha_2\mathrm{p}_y + \alpha_1\alpha_3\mathrm{p}_x\mathrm{p}_z + \alpha_2\alpha_3\mathrm{p}_y\mathrm{p}_z + \alpha_3\alpha_3\mathrm{p}_z\mathrm{p}_z + \mathrm{m}_0\mathrm{c}\alpha_4\alpha_3\mathrm{p}_z + \mathrm{m}_0\mathrm{c}\alpha_1\alpha_4\mathrm{p}_x + \mathrm{m}_0\mathrm{c}\alpha_2\alpha_4\mathrm{p}_y + \mathrm{m}_0\mathrm{c}\alpha_3\alpha_4\mathrm{p}_z + \mathrm{m}_0\mathrm{c}\mathrm{m}_0\mathrm{c}\alpha_4\alpha_4. \quad 34.6$$

For 6 to be equal to 3, you must meet the following requirements:

$$i = k \to \alpha_i^2 = \alpha_k^2 = 1 \quad 34.7$$

$$i \neq k \to \alpha_i\alpha_k + \alpha_k\alpha_i = \mathrm{zero} \quad 34.8$$

Breaking down product 6 into two equations we get 9 and 15:

$$\mathrm{p}^2 = \alpha_1\alpha_1\mathrm{p}_x\mathrm{p}_x + \alpha_2\alpha_2\mathrm{p}_y\mathrm{p}_y + \alpha_3\alpha_3\mathrm{p}_z\mathrm{p}_z + (\alpha_2\alpha_1 + \alpha_1\alpha_2)\mathrm{p}_x\mathrm{p}_y + (\alpha_3\alpha_1 + \alpha_1\alpha_3)\mathrm{p}_x\mathrm{p}_z + (\alpha_3\alpha_2 + \alpha_2\alpha_3)\mathrm{p}_y\mathrm{p}_z. \quad 34.9$$

In this case, if the matrices that represent the α_k are in accordance with 7 and 8, we will have:

$$\alpha_2\alpha_1 + \alpha_1\alpha_2 = zero \quad 34.10$$

$$\alpha_3\alpha_1 + \alpha_1\alpha_3 = zero \quad 34.11$$

$$\alpha_3\alpha_2 + \alpha_2\alpha_3 = zero \quad 34.12$$

$$\alpha_1\alpha_1 = \alpha_2\alpha_2 = \alpha_3\alpha_3 = 1 \quad 34.13$$

Com isso resulta de 9:

$$\mathrm{p}^2 = \mathrm{p}_x^2 + \mathrm{p}_y^2 + \mathrm{p}_z^2 \quad 34.14$$

The rest of product 6 is:

$$\mathrm{m}_0^2\mathrm{c}^2 = (\alpha_4\alpha_1 + \alpha_1\alpha_4)\mathrm{p}_x\mathrm{m}_0\mathrm{c} + (\alpha_4\alpha_2 + \alpha_2\alpha_4)\mathrm{p}_y\mathrm{m}_0\mathrm{c} + (\alpha_4\alpha_3 + \alpha_3\alpha_4)\mathrm{p}_z\mathrm{m}_0\mathrm{c} + \alpha_4\alpha_4\mathrm{m}_0\mathrm{c}\mathrm{m}_0\mathrm{c}. \quad 34.15$$

In this case, if the matrices that represent the α_k are in accordance with 7 and 8, we will have:

$$\alpha_4\alpha_1 + \alpha_1\alpha_4 = zero \quad 34.16$$

$$\alpha_4\alpha_2 + \alpha_2\alpha_4 = zero \quad 34.17$$

$$\alpha_4\alpha_3 + \alpha_3\alpha_4 = zero \quad 34.18$$

$$\alpha_4\alpha_4 = 1 \quad 34.19$$

This results in 15:

$$m_0^2c^2 = m_0cm_0c \quad 34.20$$

And the final result is 6 equal to 3.

The so-called Dirac α_k matrices are:

$$\alpha_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad 34.21$$

$$\alpha_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \quad 34.22$$

$$\alpha_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad 34.23$$

$$\alpha_4 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad 34.24$$

Let's do operations 10 to 13

$$\alpha_2\alpha_1 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \quad 34.25$$

$$\alpha_1\alpha_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \quad 34.26$$

$$\alpha_2\alpha_1 + \alpha_1\alpha_2 = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} + \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad 34.27$$

$$\alpha_3\alpha_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad 34.28$$

$$\alpha_1\alpha_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad 34.29$$

$$\alpha_3\alpha_1 + \alpha_1\alpha_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad 34.30$$

$$\alpha_3\alpha_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} \quad 34.31$$

$$\alpha_2\alpha_3 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix} \qquad 34.32$$

$$\alpha_3\alpha_2 + \alpha_2\alpha_3 = \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix} + \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 34.33$$

$$\alpha_1\alpha_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 34.34$$

$$\alpha_2\alpha_2 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 34.35$$

$$\alpha_3\alpha_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 34.36$$

And the requirements from 10 to 13 are fulfilled:

Let's do operations 16 to 19:

$$\alpha_4\alpha_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad 34.37$$

$$\alpha_1\alpha_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad 34.38$$

$$\alpha_4\alpha_1 + \alpha_1\alpha_4 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 34.39$$

$$\alpha_4\alpha_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \qquad 34.40$$

$$\alpha_2\alpha_4 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \qquad 34.41$$

$$\alpha_4\alpha_2 + \alpha_2\alpha_4 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad 34.42$$

$$\alpha_4\alpha_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad 34.43$$

$$\alpha_3\alpha_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad 34.44$$

$$\alpha_4\alpha_3+\alpha_3\alpha_4=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \quad 34.45$$

$$\alpha_4\alpha_4=\begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix} \quad 34.46$$

And the requirements from 16 to 19 are fulfilled:

Therefore, applying requirements 7 and 8 on the product results in 4x5 = 3:

$$\frac{E^2}{c^2}=p_x^2+p_y^2+p_z^2+m_0^2c^2=\left\{\frac{E}{c}=\alpha_1p_x+\alpha_2p_y+\alpha_3p_z+\alpha_4m_0c\right\}x\left\{\frac{E}{c}=\alpha_1p_x+\alpha_2p_y+\alpha_3p_z+\alpha_4m_0c\right\} \quad 34.47$$

In matrix form the 4x5 product is equal to:

$$\frac{E^2}{c^2}=[p_x\quad p_y\quad p_z\quad m_0c]\begin{bmatrix}\alpha_1\alpha_1&\alpha_1\alpha_2&\alpha_1\alpha_3&\alpha_1\alpha_4\\\alpha_2\alpha_1&\alpha_2\alpha_2&\alpha_2\alpha_3&\alpha_2\alpha_4\\\alpha_3\alpha_1&\alpha_3\alpha_2&\alpha_3\alpha_3&\alpha_3\alpha_4\\\alpha_4\alpha_1&\alpha_4\alpha_2&\alpha_4\alpha_3&\alpha_4\alpha_4\end{bmatrix}\begin{bmatrix}p_x\\p_y\\p_z\\m_0c\end{bmatrix} \quad 34.48$$

In this we have:

$$\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{bmatrix}[\alpha_1\quad\alpha_2\quad\alpha_3\quad\alpha_4]=\begin{bmatrix}\alpha_1\alpha_1&\alpha_1\alpha_2&\alpha_1\alpha_3&\alpha_1\alpha_4\\\alpha_2\alpha_1&\alpha_2\alpha_2&\alpha_2\alpha_3&\alpha_2\alpha_4\\\alpha_3\alpha_1&\alpha_3\alpha_2&\alpha_3\alpha_3&\alpha_3\alpha_4\\\alpha_4\alpha_1&\alpha_4\alpha_2&\alpha_4\alpha_3&\alpha_4\alpha_4\end{bmatrix} \quad 34.49$$

And applying 49 out of 48 we get:

$$\frac{E^2}{c^2}=[p_x\quad p_y\quad p_z\quad m_0c]\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{bmatrix}[\alpha_1\quad\alpha_2\quad\alpha_3\quad\alpha_4]\begin{bmatrix}p_x\\p_y\\p_z\\m_0c\end{bmatrix} \quad 34.50$$

Breaking apart we get 4 or what is the same 5:

$$\frac{E}{c}=[p_x\quad p_y\quad p_z\quad m_0c]\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{bmatrix}=\alpha_1p_x+\alpha_2p_y+\alpha_3p_z+\alpha_4m_0c \quad 34.51$$

$$\frac{E}{c}=[\alpha_1\quad\alpha_2\quad\alpha_3\quad\alpha_4]\begin{bmatrix}p_x\\p_y\\p_z\\m_0c\end{bmatrix}=\alpha_1p_x+\alpha_2p_y+\alpha_3p_z+\alpha_4m_0c \quad 34.52$$

Isolating the energy in 52 we obtain:

$$E=\alpha_1cp_x+\alpha_2cp_y+\alpha_3cp_z+\alpha_4m_0c^2 \quad 34.53$$

Let's apply the matrices α_k to 53:

$$E=\alpha_1cp_x+\alpha_2cp_y+\alpha_3cp_z+\alpha_4cm_0c^2=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}cp_x+\begin{bmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&i\\0&0&-i&0\end{bmatrix}cp_y+\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}cp_z+$$
$$\begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{bmatrix}m_0c^2 \quad 34.54$$

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} cp_x + \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} cp_y + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} cp_z + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} m_0c^2 \quad 34.55$$

On this one we must write everything in matrix form:

$$\begin{bmatrix} E \\ E \\ E \\ E \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} cp_x \\ cp_x \\ cp_x \\ cp_x \end{bmatrix} + \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \begin{bmatrix} cp_y \\ cp_y \\ cp_y \\ cp_y \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} cp_z \\ cp_z \\ cp_z \\ cp_z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_0c^2 \\ m_0c^2 \\ m_0c^2 \\ m_0c^2 \end{bmatrix} \quad 34.56$$

In this replacing the following quantum operators we obtain:

$$E = H = i\hbar\frac{\partial}{\partial t} \qquad p_x = -i\hbar\frac{\partial}{\partial x} \qquad p_y = -i\hbar\frac{\partial}{\partial y} \qquad p_z = -i\hbar\frac{\partial}{\partial z} \quad 34.57$$

$$\begin{bmatrix} i\hbar\frac{\partial}{\partial t} \\ i\hbar\frac{\partial}{\partial t} \\ i\hbar\frac{\partial}{\partial t} \\ i\hbar\frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -i\hbar c\frac{\partial}{\partial x} \\ -i\hbar c\frac{\partial}{\partial x} \\ -i\hbar c\frac{\partial}{\partial x} \\ -i\hbar c\frac{\partial}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \begin{bmatrix} -i\hbar c\frac{\partial}{\partial y} \\ -i\hbar c\frac{\partial}{\partial y} \\ -i\hbar c\frac{\partial}{\partial y} \\ -i\hbar c\frac{\partial}{\partial y} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -i\hbar c\frac{\partial}{\partial z} \\ -i\hbar c\frac{\partial}{\partial z} \\ -i\hbar c\frac{\partial}{\partial z} \\ -i\hbar c\frac{\partial}{\partial z} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_0c^2 \\ m_0c^2 \\ m_0c^2 \\ m_0c^2 \end{bmatrix} \quad 34.58$$

By respectively multiplying each level by $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ we obtain:

$$\begin{bmatrix} i\hbar\frac{\partial\Psi_1}{\partial t} \\ i\hbar\frac{\partial\Psi_2}{\partial t} \\ i\hbar\frac{\partial\Psi_3}{\partial t} \\ i\hbar\frac{\partial\Psi_4}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -i\hbar c\frac{\partial\Psi_1}{\partial x} \\ -i\hbar c\frac{\partial\Psi_2}{\partial x} \\ -i\hbar c\frac{\partial\Psi_3}{\partial x} \\ -i\hbar c\frac{\partial\Psi_4}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \begin{bmatrix} -i\hbar c\frac{\partial\Psi_1}{\partial y} \\ -i\hbar c\frac{\partial\Psi_2}{\partial y} \\ -i\hbar c\frac{\partial\Psi_3}{\partial y} \\ -i\hbar c\frac{\partial\Psi_4}{\partial y} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -i\hbar c\frac{\partial\Psi_1}{\partial z} \\ -i\hbar c\frac{\partial\Psi_2}{\partial z} \\ -i\hbar c\frac{\partial\Psi_3}{\partial z} \\ -i\hbar c\frac{\partial\Psi_4}{\partial z} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_0c^2\Psi_1 \\ m_0c^2\Psi_2 \\ m_0c^2\Psi_3 \\ m_0c^2\Psi_4 \end{bmatrix} \quad 34.59$$

From this matrix product the following Dirac equations result:

$$i\hbar\frac{\partial\Psi_1}{\partial t} = -i\hbar c\left(\frac{\partial\Psi_2}{\partial x} - i\frac{\partial\Psi_2}{\partial y} + \frac{\partial\Psi_1}{\partial z}\right) - m_0c^2\Psi_4 \quad 34.60$$

$$i\hbar\frac{\partial\Psi_2}{\partial t} = -i\hbar c\left(\frac{\partial\Psi_1}{\partial x} + i\frac{\partial\Psi_1}{\partial y} - \frac{\partial\Psi_2}{\partial z}\right) + m_0c^2\Psi_3 \quad 34.61$$

$$i\hbar\frac{\partial\Psi_3}{\partial t} = -i\hbar c\left(\frac{\partial\Psi_4}{\partial x} + i\frac{\partial\Psi_4}{\partial y} + \frac{\partial\Psi_3}{\partial z}\right) + m_0c^2\Psi_2 \quad 34.62$$

$$i\hbar\frac{\partial\Psi_4}{\partial t} = -i\hbar c\left(\frac{\partial\Psi_3}{\partial x} - i\frac{\partial\Psi_3}{\partial y} - \frac{\partial\Psi_4}{\partial z}\right) - m_0c^2\Psi_1 \quad 34.63$$

Hyperbolic equations similar to Dirac's equations 60 to 63

In what follows we are always dealing with free particle$E_p = zero$.

From the 30.9 energy equation we obtain:

$$E = c\sqrt{p^2 + m_0^2c^2} \rightarrow \frac{E^2}{c^2} - p^2 = m_0^2c^2 \quad 34.64$$

$$\frac{E^2}{c^2} - p^2 = m_0^2c^2 = \left(\frac{E}{c} + p\right).\left(\frac{E}{c} - p\right) = m_0c.m_0c \quad 34.65$$

$$\frac{\left(\frac{E}{c}+p\right)}{m_0c}.\frac{\left(\frac{E}{c}-p\right)}{m_0c} = e^{\emptyset}.e^{-\emptyset} = 1 \quad 34.66$$

That broken down into two equations results:

$$\frac{E}{c} = p + m_0ce^{-\emptyset} \quad 34.67$$

$\frac{E}{c} = -p + m_0 ce^{\emptyset}$ 34.68

In these applying $p = \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z$ 34.69

Note that the hyperbolic coefficients $e^{\emptyset}$ and $e^{-\emptyset}$ already break the term $m_0^2 c^2$ into two fractions, making the coefficient α_4. unnecessary.

$\frac{E}{c} = \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + m_0 ce^{-\emptyset}$ 34.70

$\frac{E}{c} = -\alpha_1 p_x - \alpha_2 p_y - \alpha_3 p_z + m_0 ce^{\emptyset}$ 34.71

The product of both should result in:

$\frac{E^2}{c^2} = p^2 + m_0^2 c^2 = p_x^2 + p_y^2 + p_z^2 + m_0^2 c^2$ 34.72

Let's make the 70x71 product:

$$\frac{E^2}{c^2} = -\alpha_1\alpha_1 p_x p_x - \alpha_2\alpha_1 p_x p_y - \alpha_3\alpha_1 p_x p_z + m_0 ce^{\emptyset}\alpha_1 p_x - \alpha_1\alpha_2 p_x p_y - \alpha_2\alpha_2 p_y p_y - \alpha_3\alpha_2 p_y p_z + m_0 ce^{\emptyset}\alpha_2 p_y - \alpha_1\alpha_3 p_x p_z - \alpha_2\alpha_3 p_y p_z - \alpha_3\alpha_3 p_z p_z + m_0 ce^{\emptyset}\alpha_3 p_z - m_0 ce^{-\emptyset}\alpha_1 p_x - m_0 ce^{-\emptyset}\alpha_2 p_y - m_0 ce^{-\emptyset}\alpha_3 p_z + m_0 ce^{\emptyset} m_0 ce^{-\emptyset}. \quad 34.73$$

Breaking down the product 73 into two equations X and Y we get 74 and 82:

$$X = -\alpha_1\alpha_1 p_x p_x - \alpha_2\alpha_2 p_y p_y - \alpha_3\alpha_3 p_z p_z - (\alpha_2\alpha_1 + \alpha_1\alpha_2) p_x p_y - (\alpha_3\alpha_1 + \alpha_1\alpha_3) p_x p_z - (\alpha_3\alpha_2 + \alpha_2\alpha_3) p_y p_z. \quad 34.74$$

The remainder Y of product 73 is:

$Y = m_0 c\left(e^{\emptyset} - e^{-\emptyset}\right)\left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z\right) + m_0 ce^{\emptyset} m_0 ce^{-\emptyset}.$ 34.75

From 33.11 we get:

$sh\emptyset = \frac{e^{\emptyset} - e^{-\emptyset}}{2} = \frac{cp}{m_0 c^2} \to 2p = m_0 c\left(e^{\emptyset} - e^{-\emptyset}\right)$ 34.76

In this applying 69 we obtain:

$m_0 c\left(e^{\emptyset} - e^{-\emptyset}\right) = 2p = 2\left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z\right)$ 34.77

Applying 77 out of 75 we obtain:

$Y = 2\left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z\right)\left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z\right) + m_0 ce^{\emptyset} m_0 ce^{-\emptyset}.$ 34.78

Making the product we get:

$$Y = 2\left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z\right)\left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z\right) + m_0 ce^{\emptyset} m_0 ce^{-\emptyset} = 2\begin{bmatrix}\alpha_1\alpha_1 p_x p_x + \alpha_2\alpha_2 p_y p_y + \alpha_3\alpha_3 p_z p_z + (\alpha_2\alpha_1 + \alpha_1\alpha_2) p_x p_y + \\ (\alpha_3\alpha_1 + \alpha_1\alpha_3) p_x p_z + (\alpha_3\alpha_2 + \alpha_2\alpha_3) p_y p_z\end{bmatrix} + m_0 ce^{\emptyset} m_0 ce^{-\emptyset}. \quad 34.79$$

$$Y = 2\alpha_1\alpha_1 p_x p_x + 2\alpha_2\alpha_2 p_y p_y + 2\alpha_3\alpha_3 p_z p_z + 2(\alpha_2\alpha_1 + \alpha_1\alpha_2) p_x p_y + 2(\alpha_3\alpha_1 + \alpha_1\alpha_3) p_x p_z + 2(\alpha_3\alpha_2 + \alpha_2\alpha_3) p_y p_z + m_0 ce^{\emptyset} m_0 ce^{-\emptyset}. \quad 34.80$$

Adding 80 and 74 we get:

$$\frac{E^2}{c^2} = X + Y = -\alpha_1\alpha_1 p_x p_x - \alpha_2\alpha_2 p_y p_y - \alpha_3\alpha_3 p_z p_z - (\alpha_2\alpha_1 + \alpha_1\alpha_2) p_x p_y - (\alpha_3\alpha_1 + \alpha_1\alpha_3) p_x p_z - (\alpha_3\alpha_2 + \alpha_2\alpha_3) p_y p_z + 2\alpha_1\alpha_1 p_x p_x + 2\alpha_2\alpha_2 p_y p_y + 2\alpha_3\alpha_3 p_z p_z + 2(\alpha_2\alpha_1 + \alpha_1\alpha_2) p_x p_y + 2(\alpha_3\alpha_1 + \alpha_1\alpha_3) p_x p_z + 2(\alpha_3\alpha_2 + \alpha_2\alpha_3) p_y p_z + m_0 ce^{\emptyset} m_0 ce^{-\emptyset} \quad 34.81$$

Simply put, we get:

$$\frac{E^2}{c^2} = \alpha_1\alpha_1 p_x p_x + \alpha_2\alpha_2 p_y p_y + \alpha_3\alpha_3 p_z p_z + (\alpha_2\alpha_1 + \alpha_1\alpha_2)p_x p_y + (\alpha_3\alpha_1 + \alpha_1\alpha_3)p_x p_z + (\alpha_3\alpha_2 + \alpha_2\alpha_3)p_y p_z + m_0 c e^{\emptyset} m_0 c e^{-\emptyset} \quad 34.82$$

For 82 to be equal to 72, you must meet the following requirements:

$$i = k \to \alpha_i^2 = \alpha_k^2 = 1 \quad 34.83$$

$$i \neq k \to \alpha_i\alpha_k + \alpha_k\alpha_i = \text{zero} \quad 34.84$$

To meet the requirements of 83 and 84 we must have:

$$\alpha_2\alpha_1 + \alpha_1\alpha_2 = zero \quad 34.85$$

$$\alpha_3\alpha_1 + \alpha_1\alpha_3 = zero \quad 34.86$$

$$\alpha_3\alpha_2 + \alpha_2\alpha_3 = zero \quad 34.87$$

$$\alpha_1\alpha_1 = \alpha_2\alpha_2 = \alpha_3\alpha_3 = 1 \quad 34.88$$

That makes 82 equal to 72:

$$\frac{E^2}{c^2} = p^2 + m_0^2c^2 = p_x^2 + p_y^2 + p_z^2 + m_0^2c^2 \quad 34.89$$

The so-called α_k Pauli matrices are:

$$\alpha_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad 34.90$$

$$\alpha_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad 34.91$$

$$\alpha_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad 34.92$$

Let's do the operations from 85 to 88:

$$\alpha_2\alpha_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \quad 34.93$$

$$\alpha_1\alpha_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad 34.94$$

$$\alpha_2\alpha_1 + \alpha_1\alpha_2 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} + \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad 34.95$$

$$\alpha_3\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad 34.96$$

$$\alpha_1\alpha_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad 34.97$$

$$\alpha_3\alpha_1 + \alpha_1\alpha_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad 34.98$$

$$\alpha_3\alpha_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \quad 34.99$$

$$\alpha_2\alpha_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad 34.100$$

$$\alpha_3\alpha_2 + \alpha_2\alpha_3 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} + \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad 34.101$$

$$\alpha_1\alpha_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad 34.102$$

$$\alpha_2\alpha_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad 34.103$$

$$\alpha_3\alpha_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad 34.104$$

Therefore, requirements 83 and 84 are met and we have:

$$\frac{E^2}{c^2} = p^2 + m_0^2c^2 = p_x^2 + p_y^2 + p_z^2 + m_0^2c^2 = \left\{\frac{E}{c} = \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + m_0 c e^{-\emptyset}\right\} x \left\{\frac{E}{c} = -\alpha_1 p_x - \alpha_2 p_y - \alpha_3 p_z + m_0 c e^{\emptyset}\right\} \quad 34.105$$

Isolating energy at 105 we obtain:

$$E = \alpha_1 c p_x + \alpha_2 c p_y + \alpha_3 c p_z + m_0 c^2 e^{-\emptyset} \quad 34.106$$

$$E = -\alpha_1 c p_x - \alpha_2 c p_y - \alpha_3 c p_z + m_0 c^2 e^{\emptyset} \quad 34.107$$

Let's apply the matrices α_k and the matrix I to 106 and 107:

$$E = \alpha_1 c p_x + \alpha_2 c p_y + \alpha_3 c p_z + m_0 c^2 e^{-\emptyset} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} c p_x + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} c p_y + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} c p_z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} m_0 c^2 e^{-\emptyset} \quad 34.108$$

$$E = -\alpha_1 c p_x - \alpha_2 c p_y - \alpha_3 c p_z + m_0 c^2 e^{\emptyset} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} c p_x - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} c p_y - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} c p_z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} m_0 c^2 e^{\emptyset} \quad 34.109$$

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} c p_x + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} c p_y + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} c p_z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} m_0 c^2 e^{-\emptyset} \quad 34.110$$

$$E = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} c p_x - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} c p_y - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} c p_z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} m_0 c^2 e^{\emptyset} \quad 34.111$$

On these we must write everything in matrix form:

$$\begin{bmatrix} E \\ E \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c p_x \\ c p_x \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} c p_y \\ c p_y \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} c p_z \\ c p_z \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m_0 c^2 e^{-\emptyset} \\ m_0 c^2 e^{-\emptyset} \end{bmatrix} \quad 34.112$$

$$\begin{bmatrix} E \\ E \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c p_x \\ c p_x \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} c p_y \\ c p_y \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} c p_z \\ c p_z \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m_0 c^2 e^{\emptyset} \\ m_0 c^2 e^{\emptyset} \end{bmatrix} \quad 34.113$$

In this replacing the following quantum operators we obtain:

$$E = H = i\hbar\frac{\partial}{\partial t} \qquad p_x = -i\hbar\frac{\partial}{\partial x} \qquad p_y = -i\hbar\frac{\partial}{\partial y} \qquad p_z = -i\hbar\frac{\partial}{\partial z} \quad 34.114$$

$$\begin{bmatrix} i\hbar\frac{\partial}{\partial t} \\ i\hbar\frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial}{\partial x} \\ -i\hbar c\frac{\partial}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial}{\partial y} \\ -i\hbar c\frac{\partial}{\partial y} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial}{\partial z} \\ -i\hbar c\frac{\partial}{\partial z} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m_0 c^2 e^{-\emptyset} \\ m_0 c^2 e^{-\emptyset} \end{bmatrix} \quad 34.115$$

$$\begin{bmatrix} i\hbar\frac{\partial}{\partial t} \\ i\hbar\frac{\partial}{\partial t} \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial}{\partial x} \\ -i\hbar c\frac{\partial}{\partial x} \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial}{\partial y} \\ -i\hbar c\frac{\partial}{\partial y} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial}{\partial z} \\ -i\hbar c\frac{\partial}{\partial z} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m_0 c^2 e^{\emptyset} \\ m_0 c^2 e^{\emptyset} \end{bmatrix} \quad 34.116$$

By respectively multiplying each level by $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ we obtain:

$$\begin{bmatrix} i\hbar\frac{\partial\Psi_1}{\partial t} \\ i\hbar\frac{\partial\Psi_2}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial\Psi_1}{\partial x} \\ -i\hbar c\frac{\partial\Psi_2}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial\Psi_1}{\partial y} \\ -i\hbar c\frac{\partial\Psi_2}{\partial y} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial\Psi_1}{\partial z} \\ -i\hbar c\frac{\partial\Psi_2}{\partial z} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m_0 c^2 e^{-\emptyset}\Psi_1 \\ m_0 c^2 e^{-\emptyset}\Psi_2 \end{bmatrix} \quad 34.117$$

$$\begin{bmatrix} i\hbar\frac{\partial\Psi_3}{\partial t} \\ i\hbar\frac{\partial\Psi_4}{\partial t} \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial\Psi_3}{\partial x} \\ -i\hbar c\frac{\partial\Psi_4}{\partial x} \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial\Psi_3}{\partial y} \\ -i\hbar c\frac{\partial\Psi_4}{\partial y} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -i\hbar c\frac{\partial\Psi_3}{\partial z} \\ -i\hbar c\frac{\partial\Psi_4}{\partial z} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} m_0 c^2 e^{\emptyset}\Psi_3 \\ m_0 c^2 e^{\emptyset}\Psi_4 \end{bmatrix} \quad 34.118$$

These matrix products result in the following hyperbolic equations:

$$i\hbar\frac{\partial\Psi_1}{\partial t} = -i\hbar c\left(\frac{\partial\Psi_2}{\partial x} - i\frac{\partial\Psi_2}{\partial y} + \frac{\partial\Psi_1}{\partial z}\right) + m_0c^2e^{-\emptyset}\Psi_1 \quad 34.119$$

$$i\hbar\frac{\partial\Psi_2}{\partial t} = -i\hbar c\left(\frac{\partial\Psi_1}{\partial x} + i\frac{\partial\Psi_1}{\partial y} - \frac{\partial\Psi_2}{\partial z}\right) + m_0c^2e^{-\emptyset}\Psi_2 \quad 34.120$$

$$i\hbar\frac{\partial\Psi_3}{\partial t} = -i\hbar c\left(-\frac{\partial\Psi_4}{\partial x} + i\frac{\partial\Psi_4}{\partial y} - \frac{\partial\Psi_3}{\partial z}\right) + m_0c^2e^{\emptyset}\Psi_3 \quad 34.121$$

$$i\hbar\frac{\partial\Psi_4}{\partial t} = -i\hbar c\left(-\frac{\partial\Psi_3}{\partial x} - i\frac{\partial\Psi_3}{\partial y} + \frac{\partial\Psi_4}{\partial z}\right) + m_0c^2e^{\emptyset}\Psi_4 \quad 34.122$$

Let us consider two functions $f(\emptyset) = \eta$ and $g(\emptyset) = \mu$ inversely proportional in the form:

$$f(\emptyset).g(\emptyset) = \eta.\mu = 1 \quad 35.1$$

That applied to the unitary hyperbola $x^2 - y^2 = 1$ results:

$$x^2 - y^2 = (x+y)(x-y) = \eta.\mu = 1 \quad 35.2$$

Breaking down 2 we can define the hyperbolic cosine $x = ch\emptyset$ and the hyperbolic sine $y = sh\emptyset$ in the following form:

$$\begin{cases} x + y = \eta \rightarrow x = ch\emptyset = \frac{1}{2}(\eta + \mu) \rightarrow A \\ x - y = \mu \rightarrow y = sh\emptyset = \frac{1}{2}(\eta - \mu) \rightarrow B \end{cases} \quad 35.3$$

Where we add A+B to get the $ch\emptyset$ and subtract A-B to get the $sh\emptyset$.

The hyperbolic cosine $x = ch\emptyset$ and hyperbolic sine $y = sh\emptyset$ functions are the fundamental hyperbolic functions.

Applying $x = ch\emptyset$ and $y = sh\emptyset$ to the unitary hyperbola $x^2 - y^2 = 1$ we obtain the inverse functions:

$$x^2 - y^2 = ch^2\emptyset - sh^2\emptyset = \left[\frac{1}{2}(\eta + \mu)\right]^2 - \left[\frac{1}{2}(\eta - \mu)\right]^2 \quad 35.4$$

$$x^2 - y^2 = ch^2\emptyset - sh^2\emptyset = \frac{1}{4}\eta^2 + \frac{1}{4}2\eta\mu + \frac{1}{4}\mu^2 - \frac{1}{4}\eta^2 + \frac{1}{4}2\eta\mu - \frac{1}{4}\mu^2$$

$$x^2 - y^2 = ch^2\emptyset - sh^2\emptyset = \frac{1}{4}2\eta\mu + \frac{1}{4}2\eta\mu = \eta.\mu = 1$$

$$x^2 - y^2 = ch^2\emptyset - sh^2\emptyset = \eta.\mu = 1 \quad 35.5$$

The sum of $ch\emptyset$ and $sh\emptyset$ results in η and the subtraction of $ch\emptyset$ and $sh\emptyset$ results in μ:

$$ch\emptyset + sh\emptyset = \frac{1}{2}(\eta + \mu) + \frac{1}{2}(\eta - \mu) \quad 35.6$$

$$ch\emptyset + sh\emptyset = \frac{1}{2}\eta + \frac{1}{2}\mu + \frac{1}{2}\eta - \frac{1}{2}\mu = \eta$$

$$ch\emptyset + sh\emptyset = \eta \quad 35.7$$

$$ch\emptyset - sh\emptyset = \frac{1}{2}(\eta + \mu) - \frac{1}{2}(\eta - \mu)$$

$$ch\emptyset - sh\emptyset = \frac{1}{2}\eta + \frac{1}{2}\mu - \frac{1}{2}\eta + \frac{1}{2}\mu = \mu$$

$$ch\emptyset - sh\emptyset = \mu \quad 35.8$$

So for two functions to be hyperbolic it is only necessary that they are inversely proportional in the form $f(\emptyset).g(\emptyset) = \eta.\mu = 1$.

We can construct with the unit hyperbola $x^2 - y^2 = 1$ a right triangle described as follows:

$$x^2 - y^2 = 1 \rightarrow x^2 = y^2 + 1^2 = h^2 = a^2 + b^2. \quad 35.9$$

In this triangle we have the hypotenuse h equal to x, the leg “a” equal to y and the leg b equal to 1.

With 3 and 9 we get:

$$h = x = ch\emptyset = \frac{1}{2}(\eta + \mu) \qquad a = y = sh\emptyset = \frac{1}{2}(\eta - \mu) \qquad b = 1 \quad 35.10$$

$$h + a = ch\emptyset + sh\emptyset = \eta \qquad h - a = ch\emptyset - sh\emptyset = \mu \quad 35.11$$

We can define the following trigonometric functions on this triangle:

$$a = h.\cos\alpha \qquad b = h.\text{sen}\alpha = 1 \rightarrow h = \frac{1}{\text{sen}\alpha} \qquad a = h.\cos\alpha = \frac{\cos\alpha}{\text{sen}\alpha} \qquad 35.12$$

$$h^2 = a^2 + b^2 \rightarrow h^2 = (h.\cos\alpha)^2 + (h.\text{sen}\alpha)^2 \rightarrow \text{sen}^2\alpha + \cos^2\alpha = 1 \qquad 35.13$$

From 10 and 12 we obtain the relations between the hyperbolic functions and the trigonometric functions:

$$h = \text{ch}\emptyset = \frac{1}{\text{sen}\alpha} \qquad a = \text{sh}\emptyset = \frac{\cos\alpha}{\text{sen}\alpha} \qquad 35.14$$

With 11 and 14 we get:

$$\eta = h + a = \frac{1}{\text{sen}\alpha} + \frac{\cos\alpha}{\text{sen}\alpha} = \frac{1+\cos\alpha}{\text{sen}\alpha} = \frac{\text{sen}\alpha}{1-\cos\alpha} \qquad \frac{1}{\mu} \qquad 35.15$$

$$\mu = h - a = \frac{1}{\text{sen}\alpha} - \frac{\cos\alpha}{\text{sen}\alpha} = \frac{1-\cos\alpha}{\text{sen}\alpha} = \frac{\text{sen}\alpha}{1+\cos\alpha} \qquad \mu = \frac{1}{\eta} \qquad 35.16$$

$$\eta^2 = \left(\frac{1+\cos\alpha}{\text{sen}\alpha}\right)\left(\frac{\text{sen}\alpha}{1-\cos\alpha}\right) = \frac{1+\cos\alpha}{1-\cos\alpha} \rightarrow \eta = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} \qquad \eta.\mu = 1 \qquad 35.17$$

$$\mu^2 = \left(\frac{1-\cos\alpha}{\text{sen}\alpha}\right)\left(\frac{\text{sen}\alpha}{1+\cos\alpha}\right) = \frac{1-\cos\alpha}{1+\cos\alpha} \rightarrow \mu = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \qquad \eta.\mu = 1 \qquad 35.18$$

From trigonometry we have:

$$\text{tg}\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \qquad 35.19$$

This applied in 17 and 18 results:

$$\eta = \frac{1}{\text{tg}\left(\frac{\alpha}{2}\right)} = \sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}} \qquad \eta.\mu = 1 \qquad 35.20$$

$$\mu = \text{tg}\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \qquad \eta.\mu = 1 \qquad 35.21$$

In §17 we define the proportion factors η and μ as:

$$\begin{cases} x' - ct' = \eta(x - ct) \text{ A} \\ x' + ct' = \mu(x + ct) \text{ B} \end{cases} \rightarrow 17.04 \qquad \eta.\mu = 1 \qquad 35.22$$

Equations 22 and their inverses are:

$$\begin{cases} x - ct = \mu(x' - ct') \quad \text{C} \\ x + ct = \eta(x' + ct') \quad \text{D} \end{cases} \qquad \begin{cases} x' - ct' = \eta(x - ct) \quad \text{A} \\ x' + ct' = \mu(x + ct) \quad \text{B} \end{cases} 17.04 \qquad \eta.\mu = 1 \qquad 35.23$$

Applying 20 and 21 to 23 we get:

$$\begin{cases} x - ct = \text{tg}\left(\frac{\alpha}{2}\right)(x' - ct') \;\; \text{C} \\ x + ct = \frac{1}{\text{tg}\left(\frac{\alpha}{2}\right)}(x' + ct') \quad \text{D} \end{cases} \qquad \begin{cases} x' - ct' = \frac{1}{\text{tg}\left(\frac{\alpha}{2}\right)}(x - ct) \quad \text{A} \\ x' + ct' = \text{tg}\left(\frac{\alpha}{2}\right)(x + ct) \quad \text{B} \end{cases} \qquad 35.24$$

$$\text{tg}\left(\frac{\alpha}{2}\right) = \frac{x-ct}{x'-ct'} \qquad \text{tg}\left(\frac{\alpha}{2}\right) = \frac{x'+ct'}{x+ct} \qquad 35.25$$

$$\text{tg}\left(\frac{\alpha}{2}\right) = \frac{x-ct}{x'-ct'} \qquad \text{tg}\left(\frac{\alpha}{2}\right) = \frac{x'+ct'}{x+ct} \qquad 35.26$$

$$\text{tg}\left(\frac{\alpha}{2}\right) = \frac{x-ct}{x'-ct'} = \frac{x'+ct'}{x+ct} \qquad 35.27$$

In 27 we have the description of two right triangles with the same angles.

In the following table according to 23 we have the geometry that describes 27.

The Geometry of Transformations by Hendrik Lorentz (GTHL)						
Axle $X \perp Y$	Coordinate X	Coordinate Y	Distance between points of X axis		Distance between points of Y axis	
Ponto	Observer O	Observer O'				
0	zero	zero	$X0 \leftrightarrow X1$	$x - ct$	$X0 \leftrightarrow Y1$	$x' - ct'$
1	$x - ct = \mu(x' - ct')$	$x' - ct' = \eta(x - ct)$	$X1 \leftrightarrow X2$	ct	$Y1 \leftrightarrow Y2$	ct'
2	x	x'	$X2 \leftrightarrow X3$	ct	$Y2 \leftrightarrow Y3$	ct'
3	$x + ct = \eta(x' + ct')$	$x' + ct' = \mu(x + ct)$				
The X and Y axes are perpendicular X⊥Y.						
Observer O is on the X axis and Observer O' is on the Y axis.						
The line joining $X1 \leftrightarrow Y1$ makes angle $\alpha/2$ with the Y axis.						
The line joining $X3 \leftrightarrow Y3$ makes angle $\alpha/2$ with the X axis.						

Doing on 23 the additions (C+D) and (A+B) and the subtractions (D-C) and (B-A) yields the H. Lorentz Transforms of 28 in primary form without any consideration of the propagation of a light ray. The same is obtained from the GTHL table if, for both coordinates, we average the sum of point 1 with point 3 or the average of the subtraction of point 3 minus point 1.

$$\begin{cases} x - ct = \mu(x' - ct') & C \\ x + ct = \eta(x' + ct') & D \end{cases} \qquad \begin{cases} x' - ct' = \eta(x - ct) & A \\ x' + ct' = \mu(x + ct) & B \end{cases} \quad 17.04 \qquad \eta.\mu = 1 \qquad 35.23$$

$$\begin{cases} x = \frac{1}{2}[\eta(x' + ct') + \mu(x' - ct')] \to x > ct \to \ A \ x' = \frac{1}{2}[\mu(x + ct) + \eta(x - ct)] \to x' > ct' \to \ C \\ ct = \frac{1}{2}[\eta(x' + ct') - \mu(x' - ct')] \to ct < x \to B \ ct' = \frac{1}{2}[\mu(x + ct) - \eta(x - ct)] \to ct' < x' \to \ D \end{cases} \qquad 35.28$$

In the Lorentz Transforms of 28 for both observers space is greater than time. Consequently space propagates at a speed that is greater than the speed of propagation of time which propagates at the speed of light.

"Although nobody can return behind and perform a new beginning,
any one can begin now and create a new end"
(Chico Xavier)

Bibliography

[1] Alonso & Finn, Vol. I e II, Ed. Edgard Blücher LTDA, 1972.
[2] Robert Resnik, Introduçao à Relatividade Especial, Ed. Univ. de S. Paulo e Ed. Polígono S.A., 1971.
[3] E. Terradas Y R. Ortiz, Relatividad, Cia. Ed. Espasa-Calpe Argentina S. A., 1952.
[4] Abraham Pais, "Sutil é o Senhor..." A ciência e a vida de Albert Einstein, Ed. Nova Fronteira, 1995.
[5] Marcel Rouault, Física Atômica, Ao Livro Técnico LTDA, 1959.
[6] H. A. Lorentz, A. Einstein e H. Minkowisk, O Princípio da Relatividade, edição da Fundação Calouste Gulbenkian, 1989.
[7] L. Landau e E. Lifchitz, Teoria do Campo, Hemus – Livraria Editora LTDA.
[8] Robert Martin Eisberg, Fundamentos da Física Moderna, Ed. Guanabara Dois S.A., 1979.
[9] Max Born, Física Atômica, edição da Fundação Calouste Gulbenkian, 1986.
http://www.wbabin.net/physics/faraj7.htm
[10]Princípios de Mecânica Quântica Vladimir A. Fock.